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Tides in the Earth–Moon System

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Abstract. The goal of this paper is to present a simple explanation of tides and a simple derivation of tidal forces. Everything should be done with the use of high school mathematics only. The influence of the eccentricity of the Moon and of the Earth is also discussed.

Introduction

Changes in the sea level are very well known as tides but the explanation is not as easy as the simple statement that they are caused due to the gravity of the Moon. The understanding of tides is not easy and many students (and even teachers) have many misconceptions (which occur also in some textbooks) [1, 2]. The purpose of this paper is to explain tides in the form understandable to high school students with the use of simple mathematics and illustrative pictures.[1] In the paper we will focus on the influence of the orbital eccentricity of the Moon and of the Earth on tides.

Newton's laws

The magnitude of the force attracting two bodies with masses m and M to each other is given by the *Newton's law of gravity* (G is the gravitational constant)

$$F_g = \frac{GmM}{r^2},$$

where r is the distance between them. Is is useful to describe the gravitational field of the body with the mass M with using the gravitational acceleration (this can be also understood as the gravitational force per unit mass)—with the use of the *Newton's second law of motion* we get

$$a_g = \frac{F_g}{m} = \frac{GM}{r^2}. \tag{1}$$

In our case, the body with the mass M will be the Moon or the Sun and the body with the mass m will be the whole planet Earth or only its part. We will assume that the Earth's rotational axis is orthogonal to its orbital plane and that the Moon orbits around the Earth above the Earth's equator.[2] We will also assume that the whole Earth is covered with water.

In this article we will use the notation and values described in Table 1.

The explanation of tides

Because the gravitational field is inhomogeneous (1), the gravitational acceleration is bigger at the point A (a point on the Earth closest to the second body) than in the center O of the Earth and the gravitational acceleration in the center of the Earth is bigger than at the point C (a point on the Earth furthest from the second body). Because the Earth is in a free fall, we can subtract the gravitational acceleration from the Moon (or the Sun) in the center of the Earth from all gravitational accelerations at other points inside/on the Earth—see Fig. 1.

[1] All pictures in this paper were created in software Wolfram Mathematica 8.0.4.0 for Students [3].

[2] In fact, the Earth's axial tilt is not 0° but 23.4° and the Moon's inclination to the ecliptic is not 0° but 5.1° [4]. We will disregard these differences.

Table 1. The notation and important values (distances are measured from the center of one body to the center of another). Values were taken from [4].

gravitational constant	$G = 6.673 \cdot 10^{-11}$ N·m^2·kg^{-2}
Earth mass	$M_\oplus = 5.974 \cdot 10^{24}$ kg
Moon mass	$M_☾ = 7.349 \cdot 10^{22}$ kg
Sun mass	$M_\odot = 1.989 \cdot 10^{30}$ kg
Earth–Moon mean distance	$r_☾ = 3.844 \cdot 10^8$ m
Earth–Moon maximum distance (apogee)	$r_{☾\,\mathrm{max}} = 4.07 \cdot 10^8$ m
Earth–Moon minimum distance (perigee)	$r_{☾\,\mathrm{min}} = 3.57 \cdot 10^8$ m
Earth–Sun mean distance	$r_\odot = 1.496 \cdot 10^{11}$ m
Earth–Sun maximum distance (aphelion)	$r_{\odot\mathrm{max}} = 1.521 \cdot 10^{11}$ m
Earth–Sun minimum distance (perihelion)	$r_{\odot\mathrm{min}} = 1.471 \cdot 10^{11}$ m
Earth equatorial radius	$R_\oplus = 6.378 \cdot 10^6$ m

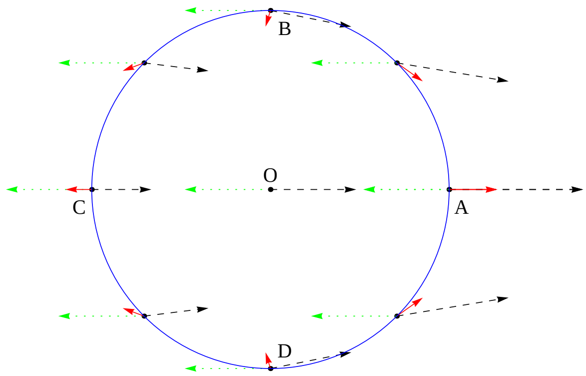

Figure 1. The origin of tidal forces. Another massive body is placed very close to the object (on the line AC closer to the point A)–in that case we can very well see the principle: from all gravitational accelerations (dashed black), (accelerations from the second body on the body in the picture) we subtract the gravitational acceleration in the center (dotted green) and after that operation with vectors we get the result–the tidal accelerations (red vectors). It is obvious that tidal accelerations are smaller than original gravitational accelerations. We can see that the picture is symmetrical along the line AC but not along BD. See Fig. 2 for a more realistic situation.

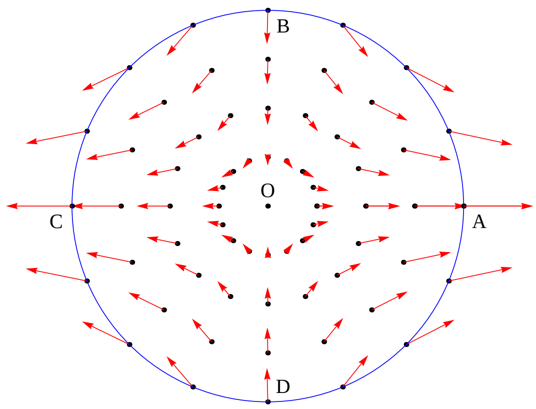

Figure 2. Tidal forces (only) from the Moon (in its mean distance from the Earth) on the Earth. The result, caused by the inhomogeneous gravitational field of the Moon, was obtained as described in Fig. 1. We see that there are two bulges–except the bulge at the point A there is also the bulge in the opposite side of the Earth. The tides in these two points are called *high tides*. Tidal forces on the Earth's surface are smallest at points B and D and such tides are called *low tides*. Tidal forces are getting smaller as we go closer to the center of the Earth. Tidal forces at points A, C head from the center O and tidal forces at points B, D head approximately to the center.

We can use a very nice analogy, stated in [5]: imagine three astronauts on one line with same masses falling toward the Moon (astronaut A is the closest one to the Moon, astronaut C the furthest one and astronaut O lies between them). Because the gravitational accelerations of these three astronauts are different, the astronaut O will see that the astronaut A is accelerating away from him toward the Moon and that the astronaut C is accelerating away from him in an opposite direction from the astronaut A. These explanations are related to the center of the Earth.

Another explanation which could help students can be: the gravitational acceleration is the biggest at the point A, smaller at the point O and smallest at the point C, so a molecule at the point A will get closer to the Moon, at the point O also closer but not as much as at the point A and at the point C it will not move—now it is related to the point C.

We can conclude: the tidal accelerations (forces per unit mass) are given as differences between gravitational accelerations on/inside the Earth and the gravitational acceleration in the center O of the Earth. The most difficult result for students' understanding is that there exists not only a bulge at the point A, but also a bulge at the point C, in the opposite side of the Earth from the Moon [1]. Because it is 3D problem, the resulting shape of a spherical body is an ellipsoid ("rugby ball").

Calculations—the magnitude of tidal forces

The role of the eccentricity

We can simply calculate tidal forces (per unit mass) at points A and C:

$$\delta a_{g☾}(\mathrm{A}) = a_{g☾}(\mathrm{A}) - a_{g☾}(\mathrm{O}) = \frac{GM_☾}{\left(r_☾ - R_\oplus\right)^2} - \frac{GM_☾}{r_☾^2} \approx 1.129 \cdot 10^{-6}\ \mathrm{N/kg},$$

$$\delta a_{g☾}(\mathrm{C}) = a_{g☾}(\mathrm{C}) - a_{g☾}(\mathrm{O}) = \frac{GM_☾}{\left(r_☾ + R_\oplus\right)^2} - \frac{GM_☾}{r_☾^2} \approx -1.075 \cdot 10^{-6}\ \mathrm{N/kg},$$

where the negative sign means that $\delta\vec{a}_g(\mathrm{C})$ has an opposite direction than $\vec{a}_g(\mathrm{O})$. We can see that $\delta a_g(\mathrm{A}) > |\delta a_g(\mathrm{C})|$ but the difference is relatively small. The average of these tidal forces is

$$\frac{\delta a_{g☾}(\mathrm{A}) + \left|\delta a_{g☾}(\mathrm{C})\right|}{2} = 1.102 \cdot 10^{-6}\ \mathrm{N/kg}. \tag{2}$$

We can do an approximate calculation:

$$\delta a_{g☾}(\mathrm{C}) = \frac{GM_☾}{\left(r_☾ - R_\oplus\right)^2} - \frac{GM_☾}{r_☾^2} = GM_☾ \frac{2r_☾ R_\oplus - R_\oplus^2}{\left(r_☾ - R_\oplus\right)^2 r_☾^2} \approx GM_☾ \frac{2r_☾ R_\oplus}{r_☾^4} = \frac{2GM_☾ R_\oplus}{r_☾^3}, \tag{3}$$

because $2r_☾ R_\oplus - R_\oplus^2 \approx 2r_☾ R_\oplus$ and $\left(r_☾ - R_\oplus\right)^2 \approx r_☾^2$.
Equation (3) leads to the result $1.101 \cdot 10^{-6}$ N/kg and when we compare that result with (2), we can see that equation (3) is a good approximation for the magnitude of tidal forces at points A and C.

Now we will calculate the tidal forces (per unit mass) at points B and D (they are exactly equal). Let's assume such Cartesian coordinate system with the origin in the center of the Earth, where the positive x-axis points to the center of the Moon. For the vector of the gravitational acceleration from the Moon in the center of the Earth we obtain

$$\vec{a}_{g☾}(\mathrm{O}) = \frac{GM_☾}{r_☾^2}(1, 0).$$

Table 2. The magnitudes of tidal forces from the Moon (left table) and the Sun (right table), (separately, which means that we take into account only tidal forces from one body). The units are $1 \cdot 10^{-6}$ N/kg.

Earth–Moon distance	mean	max	min
point A or C	1.101	0.928	1.375
point B or D	0.551	0.464	0.688

Earth–Sun distance	mean	max	min
point A or C	0.506	0.481	0.532
point B or D	0.253	0.241	0.266

The point B has coordinates $[0, R_\oplus]$, the center of the Moon $\left[r_☾, 0\right]$, so

$$\vec{a}_{g☾}(\mathrm{B}) = \frac{GM_☾}{r_☾^2 + R_\oplus^2} \frac{\left(r_☾, -R_\oplus\right)}{\sqrt{r_☾^2 + R_\oplus^2}}.$$

The norm of the resulting tidal force $\delta\vec{a}_{g☾}(\mathrm{B}) = \vec{a}_{g☾}(\mathrm{B}) - \vec{a}_{g☾}(\mathrm{O})$ is $0.550 \cdot 10^{-6}$ N/kg.

We can do a simpler calculation, because the Moon is far enough, $\delta\vec{a}_{g☾}(\mathrm{B})$ heads approximately to the center of the Earth[3]

$$\frac{R_\oplus}{r_☾} = \tan\alpha \approx \frac{\delta a_{g☾}(\mathrm{B})}{a_{g☾}(\mathrm{O})},$$

where α is the angle between the Earth–Moon line and the connecting line from the point B to the Moon.

$$\delta a_{g☾}(\mathrm{B}) = \delta a_{g☾}(\mathrm{D}) \approx a_{g☾}(\mathrm{O})\frac{R_\oplus}{r_☾} = \frac{GM_☾}{r_☾^2}\frac{R_\oplus}{r_☾} = \frac{GM_☾ R_\oplus}{r_☾^3} = \frac{\delta a_{g☾}(\mathrm{A})}{2} \approx 0.551 \cdot 10^{-6}\ \mathrm{N/kg}.$$

A summary:

$$\delta a_{g☾}(\mathrm{A}) \approx \delta a_{g☾}(\mathrm{C}) \approx \frac{2GM_☾ R_\oplus}{r_☾^3}, \qquad \delta a_{g☾}(\mathrm{B}) = \delta a_{g☾}(\mathrm{D}) \approx \frac{\delta a_{g☾}(\mathrm{A})}{2}, \tag{4}$$

$\delta\vec{a}_g$ at points A and C heads away from the center of the Earth, $\delta\vec{a}_g$ at points B and D heads to the center of the the Earth. For the Sun we obtain similar equations (just replace the symbol ☾ with $\odot$), the results are in the Table 2. In that table we take into account the orbital eccentricity of the Moon (0.0549) and of the Earth (0.0167), [3].

In the Table 2 we see that lunar tidal force can be about 16 % smaller than usual when the Moon is in the apogee and about 25 % bigger than usual when it is in the perigee. Solar tidal force can be about 5 % smaller than usual when the Earth is in the aphelion and about 5 % bigger than usual when it is in the perihelion. The lunar tidal force is 2.2 bigger than solar tide when both bodies are in their mean distances, that ratio can vary from 1.7 (the Moon is in the apogee while the Earth is in the perihelion) to 2.9 (the Moon is in the perigee while the Earth is in the aphelion).

The Moon and the Sun together

Now we will consider that both bodies—the Moon and the Sun—are acting together. At first, let both bodies lie on one line AC (Full Moon or New Moon). In that case we obtain the resulting tidal force simply by the summation of tidal forces from the Moon and the Sun. The high tide in that case is called *spring tide*. See Fig. 3 and Table 3.

[3]In fact, the angular deflection of $\delta\vec{a}_g(\mathrm{B})$ from the direction to the center of the Earth is 1.4° for the lunar tidal force and $(3.7 \cdot 10^{-3})°$ for the solar tidal force.

Table 3. Resulting tidal forces from the Moon and the Sun when both lie on line AC (Full Moon or New Moon). The units are $1 \cdot 10^{-6}$ N/kg.

distance of both bodies	mean	maximum	minimum
point A or C	1.607	1.399	1.907
point B or D	0.804	0.700	0.954

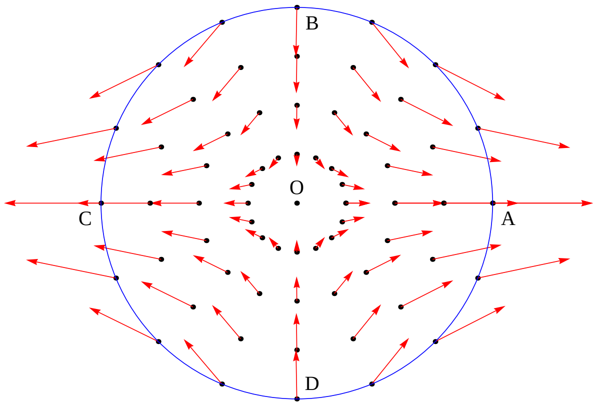

Figure 3. Spring tide. If we take into account also the Sun, we get biggest tidal forces in case of the Sun, the Earth and the Moon are in one line (New Moon or Full Moon), we don't give a specific position where the Moon or the Sun are located–whether the Moon or the Sun is closer to the point A or C–because the result is approximately the same in both situations). The values were calculated for mean values of the distances. The scale of the picture is exactly the same as in Fig. 2. Compare the magnitudes of the tidal forces with Fig. 4.

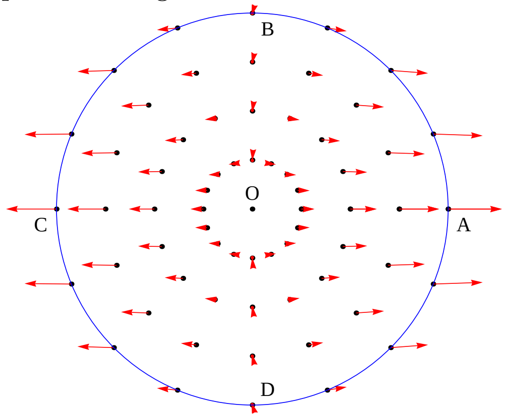

Figure 4. Neap tide. If the Moon is in the phase of the First or Last Quarter, the tidal forces are the smallest. The situation is such that the Sun lies on the line BD and the Moon on the line AC (we don't give specific positions where the Moon and the Sun are located–whether the Moon is closer to the point A or C and whether the Sun is closer to the point B or D–because the result is approximately the same for all configurations). The values were calculated for mean values of the distances. The scale of the picture is exactly the same as in Figs. 2 and 3.

With any other combination of distances of the Moon and the Sun we do not get bigger or smaller effects. It was only an addition of vectors and it is still valid that at points B, D are tidal forces two times smaller than at points A, C. We can see that spring tide can be about 13 % smaller than usual spring tide (when the Moon is in the apogee and the Earth in the aphelion) and about 19 % bigger (when the Moon is in the perigee and the Earth in the perihelion).[4]

[4] Such a rare configuration, when the Moon was (almost) in the perigee at the Full Moon and the Earth was (almost) in the perihelion, occured on January 4, 1912 and it even seems that it had the influence on sinking the Titanic on April 14, 1912 [6].

Table 4. Resulting tidal forces from the Moon and the Sun at the First or Last Quarter. The units are $1 \cdot 10^{-6}$ N/kg. The negative sign means that tidal forces at points B, D don't head to the center of the Earth but away from the center.

Earth–Moon distance	mean	mean	mean	max	max	max	min	min	min
Earth–Sun distance	mean	max	min	mean	max	min	mean	max	min
point A or C	0.848	0.861	0.835	0.675	0.688	0.662	1.122	1.135	1.109
point B or D	0.045	0.070	0.019	−0.042	−0.017	−0.068	0.182	0.207	0.156

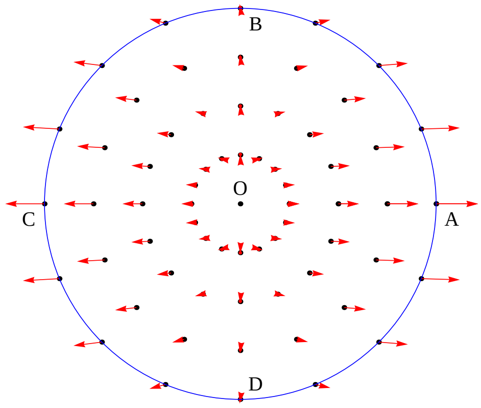

Figure 5. Tidal forces at the First (or Last) Quarter when the Moon is in the apogee and the Earth is in the perihelion (positions of these bodies are the same as described in Fig. 4). There is no high tide at points B, D (tidal forces at these points head away from the Earth). The scale of the picture is exactly the same as in the Figs. 2, 3, and 4.

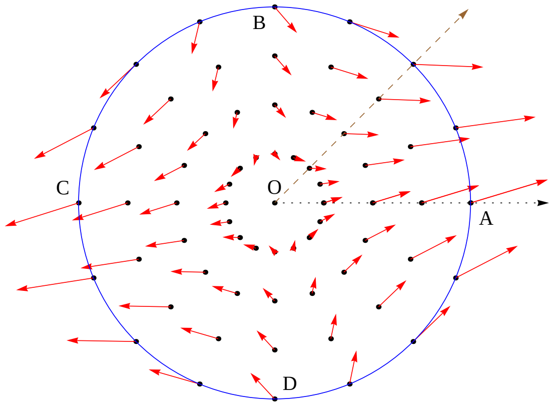

Figure 6. Tidal forces in the general configuration of the Moon and the Sun (in their mean distances). Dotted black vector heads to the Moon, dashed brown vector heads to the Sun (the angle between the Earth–Sun line and the Earth–Moon line is 45°). We can see that the tidal force at the point A doesn't head to the Moon.

Now let's consider the situation at the First or Last Quarter = all three bodies (the Moon, the Earth, the Sun) create right-angled triangle. The Moon lies on the line AC, the Sun on the line BD (again, we get approximately identical results for both positions of the Moon and for both positions of the Sun at both points A, C and also at points B, D). At all points A, B, C, D we have to subtract from the tidal force of the Moon the tidal force from the Sun. In this case we have more combinations than in the previous situation, we can get different results for different distances of the Moon and of the Sun. The high tide in that case is called *neap tide*. See Fig. 4 and Table 4.

From the Table 4 we see that the neap tide can be 22 % smaller than usual neap tide (when the Moon is in the apogee and the Earth in the perihelion) and about 34 % bigger than usual neap tide (when the Moon is in the perigee and the Earth in the aphelion). The most interesting fact in the Table 4 is that there is no high tide at points B, D when the Moon is in the apogee (at the First or Last Quarter), see Fig. 5.

We can also compare the spring and neap tide, see Tables 3 and 4, the spring tide is about 1.9 times bigger than the neap tide in usual (both bodies in their mean distances), but that ratio can vary from 1.2 to 2.9. It it also interesting that the tidal force at points B, D (when both bodies are in their minimum distances during spring tide) can be bigger than the tidal force at points A, C (for example when both bodies are in their mean distances, but there are many other possibilities).

When the Moon and the Sun are in general positions (with respect to the Earth), the derivation of the magnitude of tidal forces is difficult and we include the Figure 6 for one particular configuration of the Moon, the Sun and the Earth at least.

Conclusions

The purpose of this paper was to explain tides in the form understandable to high school students. We performed several calculations of tidal forces and we discussed the influence of the eccentricity of the Moon and of the Earth to tidal forces. We created several pictures which could help students understand the tides correctly.

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