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Determination of Topology of Offset Curves

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Abstract. We introduce a new approach to the problem of determination of the topology of curves based on the dual representation of curves using their support function. Due to this representation we can easily find critical points and recover the topology of the curve. This approach can be used to solve many problems, for example to find the topologically equivalent graph of offset curves and convolutions.

Introduction

Many problems in Computer Aided Geometric Design (CAGD), such as computation of the intersection of a rational spatial surface with a plane, are reduced to determination of a graph of an implicitly defined bivariate polynomial with rational coefficients. As is usual in CAGD we are interested in the output with which we can easily work. It is very desiderable if we can visualize it in any precision, find out the number of components or test to which component a point belongs. All this information is contained in a planar graph topologically equivalent (i.e. ambient isotopic in real plane) to the curve and whose vertices are points of the curve and edges correspond to arcs of the curve.

In this paper we present a new approach to the problem of determination of topology of a planar real algebraic curve $\mathcal{C} = \{(x, y) \in \mathcal{D} \mid f(x, y) = 0\}$, given by a bivariate polynomial with rational coefficients, i.e. $f \in \mathbb{Q}[x, y]$, in a region $\mathcal{D} = [a, b] \times [c, d] \subset \mathbb{R}^2$. Our algorithm uses the support function to describe the dual representation of the curve and on the dual representation it finds the points which corresponds to important points on the original curve.

Our algorithm can be extended to computation of topology of offset curves or convolutions of curves, because these operations correspond to simple operations on the duals curves.

The paper is divided into two parts. In the first part we summarize known results – in section 2 we sum up the previous work and the main approaches to this problem. Then follow preliminaries to our results. We recall definitions and basic properties of support function and convolutions.

The second part of this paper contains new results. In section four we describe duality between cusps and inflection points, we use their properties to compute them effectively. We also describe the way how to find extremal and boundary points. We finish this section by the algorithm. The following section is devoted to the application of this algorithm to offset curves. Finally we conclude the paper.

Previous work

Consider a vertical segment $s = \{y \in \mathbb{R} \mid y \in [c, d]\}$ moving from the left side ($x = a$) of the region to the right side ($x = b$). In every time there is a finite number of intersections of s and $\mathcal{C}$. The number of intersections can change only when $\mathcal{C}$ has a *critical point* on this x-coordinate. To ensure that the graph $\mathcal{G}$ is topologically equivalent to $\mathcal{C}$ we have to include all critical points in the region $\mathcal{D}$ among vertices of $\mathcal{G}$. Namely

- singularities of $\mathcal{C}$:
  - self-intersections
  - isolated points
  - cusps
- points of $\mathcal{C}$ with vertical tangent line and
- intersections of $\mathcal{C}$ with the boundary of the region $\mathcal{D}$.

There are two main types of algorithms. The first type uses the same principle as the Cylindrical Algebraic Decomposition algorithm (*Basu et al.* [2006], p. 159). The other approach is based on a subdivision of the given region.

Cylindrical Algebraic Decomposition based algorithms

These algorithms are usually divided into three phases and work over $\mathbb{C}$:

Phase 1: Find out x-coordinates of critical points of $\mathcal{C}$
We project the critical points on x-axis. Using sub-resultant sequence, we compute the determinant $R(x)$ of f which is an univariate polynomial with rational coefficients. Then we determine the roots of $R(x)$ and obtain the x-coordinates (x_i, $1 \leq i \leq n$) of all critical points of $\mathcal{C}$.

Phase 2: For each x_i compute intersection points of $\mathcal{C}$ and the vertical line $x = x_i$
The intersection points $P_{i,j}$ $(1 \leq j \leq m)$ have as y-coordinates the roots of the polynomial $f(x_i, y)$.

Phase 3: For every $P_{i,j}$ determine the number of branches of $\mathcal{C}$ on the left and right
Using this information we can connect the points appropriately.

The main problem of these algorithms is the Phase 2, because x-coordinates of critical points are not necessarily rational numbers and therefore the polynomials $f(x_i, y)$ have non-rational coefficients.

There are several methods to deal with this problem. *Hong* [1996] computes xy-parallel separating boxes of critical points with rational endpoints. Then he can count the branches in Phase 3 as roots of univariate polynomial with rationals coefficients.

Gonzales-Vega and Necula [2002] proposed **Phase 0: Linear change of coordinate**. The x-coordinate is transformed so that the curve is in generic position.

Definition 1. The real algebraic curve $\mathcal{C}$ is in *generic position* if it satisfies the following conditions:

- the curve $\mathcal{C}$ has no vertical asymptotes
- on every vertical line $x = \alpha$, $\alpha \in \mathbb{R}$ is at most one critical point

Obviously there are only $\binom{c}{2}$ non-generic configurations, where c is a number of critical points. Therefore the change of coordinates is always possible.

When the curve is in generic position, *Gonzales-Vega and Necula* [2002] use the *Sturm-Habicht* sequence, a suitable generalization of polynomial remainder sequence, to derive the y-coordinates of critical points (Phase 2) as rational functions of their x-coordinate and also to deduce the multiplicity of the considered critical point.

Another solution was given by *Seidel and Wolpert* [2005] - they project critical points to three axes x, y and a random one. From these projections they can recover xy-parallel boxes with rational endpoints which separate the critical points.

Eigenwilling et al. [2007] give the Bitstream Descartes algorithm (a variant of interval Descartes algorithm) as an efficient algorithm to isolate roots of a polynomial with non-rational coefficients.

In contrast to all above algorithms *Cheng et al.* [2009] replace the Sturm-Habicht sequence with a Gröbner basis and rational univariate representation, which ensure that we avoid working with polynomials with non-rational coefficients even in non-generic position.

A subdivision based algorithm

The only certified algorithm (i.e. one which gives the correct output for every input) based on subdivision is *Alberti et al.* [2008]. This algorithm subdivides the region $\mathcal{D}$ into *regular regions* (the curve is smooth inside) and *regions with singular points*, which can be made sufficiently small. The topology inside the regions containing a singular point is recovered from the information on the boundary using the topological degree.

Preliminaries

In this section we recall the definitions of the support function and the convolution and we study their basic properties. These notions are defined over $\mathbb{R}$.

Support function

The support function representation of curves and surfaces is a classical tool of convex geometry. The support function assigns to a unit normal vector the distance between the origin and the tangent line of the curve. The curve can be recovered from the support function as the envelope of the set of all tangent lines. For more details see *Aigner et al.* [2009].

Definition 2. Let $\mathcal{C}$ be a curve in projective plane. The *dual representation of* $\mathcal{C}$ is a set in the dual projective plane consisting of tangent lines of $\mathcal{C}$.

The dual representation $\mathcal{H}$ is given by $g(h, \mathbf{n}) = 0$, where g is a homogenous polynomial in a parameter h and the normal vector $\mathbf{n} = (n_1, n_2)$. If the normal vector has norm 1, then h is the oriented distance between the tangent line and the origin.

Definition 3. For a regular point $(h_0, \mathbf{n_0}) \subset \mathbb{R} \times \mathbb{R}^2$ of $\mathcal{H}$, we get

$$h : \mathbf{n} \to h(\mathbf{n})$$

in a certain neighborhood of $(h_0, \mathbf{n_0})$ from implicit function theorem. The restriction of h to the unit circle is called the *support function of* $\mathcal{C}$.

In the rest of this paper, where it is clear which curve we consider, h will denote the support function of the considered curve.

As proved in *Šír et al.* [2008] we can recover the curve $\mathcal{C}$ from h using the envelope formula:

$$p(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + \dot{h}(\mathbf{n})\mathbf{n}^{\perp}, \tag{1}$$

where p is the point of $\mathcal{C}$ with the unit normal vector $\mathbf{n}$, $\dot{h}$ denote the derivative with respect to the arc length and $^{\perp}$ is a sign for the clockwise rotation of $\mathbf{n}$ about the origin by the angle $\frac{\pi}{2}$.

Convolutions

Convolutions of curves have natural applications in many fields such as CAGD, CNC machining, robotics etc. The special case – convolution with a circle, called offset – deserves particular attention.

Definition 4. Let $\mathcal{A}, \mathcal{B}$ be real algebraic curves. Their *convolution* is a set

$$\mathcal{A} \star \mathcal{B} = \operatorname{cl}\{a + b \mid a \in \mathcal{A},\ b \in \mathcal{B},\ \mathbf{n}(a) \parallel \mathbf{n}(b)\},$$

where $\mathbf{n}(x)$ denotes the normal vector of the appropriate curve at the regular point x. The symbol cl is the notation for Zariski closure.

The convolution of a curve $\mathcal{A}$ with a circle of radius d is called the *offset curve to* $\mathcal{A}$ *at distance* d.

One of the advantages of the support function representation of curves is that the convolution and similar operations are very simple. See *Šír et al.* [2008] for details.

Theorem 5. *Let* $\mathcal{A}$, $\mathcal{B}$ *be real algebraic curves and* $h_{\mathcal{A}}, h_{\mathcal{B}}$ *their support functions. The support function of* $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ *satisfies* $h_{\mathcal{C}} = h_{\mathcal{A}} + h_{\mathcal{B}}$. *The support function of the offset curve* o_d *of* $\mathcal{A}$ *at the distance* d *is* $h_{o_d} = h_{\mathcal{A}} + d$.

Our approach

We present a new approach to the problem of the determination of topology of real algebraic curves that uses the support function representation to find some critical points of the curve easier.

We limit ourselves to the class of curves whose support functions are functions, i.e. we have to test that for every point of the curve is defined the support function (this assumption excludes curves with isolated points) and that for every unit normal the support function has only one value (equivallently that the curve has no inflection points).

The connection process of our algorithm is not dependent on the position of self-intersections of the curve and we can find them afterwards using another algorithm.

Cusps

From general theory (see e.g. *Walker* [1978]) the cusps on $\mathcal{C}$ correspond to inflection points in the dual representation $\mathcal{H}$. Using the envelope formula (1) we get the necessary condition for cusps

$$h(\mathbf{n}) + \ddot{h}(\mathbf{n}) = 0. \tag{2}$$

Let $\mathbf{n} = (n_1(s), n_2(s))$ be a parametrization of the unit circle by arc length s. In the following we omit the argument of the support function h, which is always $\mathbf{n}$. Using the chain rule we get

$$\dot{h} = h_{n_1}\dot{n}_1 + h_{n_2}\dot{n}_2 = -h_{n_1}n_2 + h_{n_2}n_1 \tag{3}$$

$$\begin{aligned}\ddot{h} &= h_{n_1n_1}\dot{n}_1^2 + h_{n_1n_2}\dot{n}_1\dot{n}_2 + h_{n_1}\ddot{n}_1 + h_{n_2n_2}\dot{n}_2^2 + h_{n_2n_1}\dot{n}_1\dot{n}_2 + h_{n_2}\ddot{n}_2 = \\ &= h_{n_1n_1}n_2^2 - h_{n_1n_2}n_2n_1 - h_{n_1}n_1 + h_{n_2n_2}n_1^2 - h_{n_2n_1}n_1n_2 - h_{n_2}n_2,\end{aligned} \tag{4}$$

where the dot denotes the derivative with respect to arc length s and the subscript denotes the partial derivative. The second equality in (3) and in (4) is deduced using the equality $(\dot{n}_1, \dot{n}_2) = (-n_2, n_1)$.

The partial derivatives of h can be deduced from its implicit definition. For example:

$$g_{n_1}(h(\mathbf{n}), n_1, n_2) = g_{n_1}(h, n_1, n_2) + h_{n_1} g_h(h, n_1, n_2) = 0.$$

And therefore

$$h_{n_1} = -\frac{g_{n_1}(h, n_1, n_2)}{g_h(h, n_1, n_2)}.$$

Similarly we can deduce all partial derivatives of h and substitute them into (4). This equation we substitute into (2) to get a necessary condition for cusps in variables n_1, n_2. Again, we omit the arguments of g.

$$h - \left(\frac{1}{g_h^3}\right)(n_1^2(g_h^2 g_{n_2n_2} + g_{hh}g_{n_2}^2 - 2g_h g_{hn_2} g_{n_2}) + n_2^2(g_h^2 g_{n_1n_1} + g_{hh}g_{n_1}^2 - 2g_h g_{hn_1} g_{n_1}) + \\ + 2n_1n_2(g_h g_{hn_2} g_{n_1} + g_h g_{hn_1} g_{n_2} + g_{hh} g_{n_1} g_{n_2} - g_h^2 g_{n_1n_2}) - n_1 g_h^2 g_{n_1} - n_2 g_h^2 g_{n_2}) = 0 \qquad (5)$$

Extremal points

It is also easy to find extremal points (those with tangent parallel to x-axis resp. y-axis), because their normal vector is $\mathbf{n} = (1, 0)$ resp. $\mathbf{n} = (0, 1)$. In this situation we can compute the value of support function h and use the envelope formula (1) to recover extremal points on the curve $\mathcal{C}$.

Boundary points

The boundary points of $\mathcal{C}$ are solutions of the system of equations (6) and (7) for vertical sides and (6) and (8) for horizontal sides.

$$g(h, \mathbf{n}) = 0 \qquad (6)$$

$$h(\mathbf{n})n_1 - \dot{h}(\mathbf{n})n_2 - a \text{ (resp. } b) = 0 \qquad (7)$$

$$h(\mathbf{n})n_2 + \dot{h}(\mathbf{n})n_1 - c \text{ (resp. } d) = 0 \qquad (8)$$

To solve this system, we first compute the resultant of the polynomial $g(h, \mathbf{n})$ with the condition that $\mathbf{n}$ is of unit length, $n_1^2 + n_2^2 - 1$, with respect to n_2. Then we can compute the resultant of the result and the left side of (7) resp. (8) with respect to h and then determine the roots of an univariate polynomial in n_1.

Algorithm

The support function is in general a multivalued map. We restrict ourselves to the case of curves $\mathcal{C}$, whose support function is a function. This means that every unit normal vector appears at most once on $\mathcal{C}$.

In this case the connectivity of the critical points is clear. We sort and connect points by the increasing angle of normal vector $\mathbf{n}$ of their support function representation.

Input: Real algebraic curve $\mathcal{C}$ given as a zero set of a bivariate polynomial with rational coefficients $f(x, y) \in \mathbb{Q}[x, y]$ on which every normal vector appears at most once.
Output: Approximation of the curve with linear segments.

Step 1: Find out the support function h of $\mathcal{C}$.
Result of this step is a implicit definition of $g(h, \mathbf{n}) = 0$.

Step 2: Determine the cusps, extremal points and boundary points with multiplicity.
This step proceeds as described in the first part of this section.

Step 3: Connect points by the angle of $\mathbf{n}$ running around a unit circle. Delete all segments connecting only the boundary points.

Step 4: Find the points on the original curve and connect them as in the dual space.
We find the points using envelope formula (1).

Application to Convolutions

At this time, there exists no implemented or sufficiently effective algorithm to compute the topology of convolutions. The only existing paper is *Farouki et al.* [2007]. We tried to implement this algorithm and we uncovered several implementation problems, such as solving a system of rational equations in four variables or the assumption that we are working only with exact numbers.

Our algorithm needs only a slight modification to be used to the problem to determine the topology of convolutions or offsets. The only difference is that Step 1 is performed in two phases, first we find the support function of original curve(s) and then we compute the support function of convolution or offset as described in the preliminary section.

Example

We will demonstrate our algorithm on an example: Determine the topology of the offset to the curve defined by $f(x, y) = y - x^4$ at the distance -1 within the box $[-0.5, 0.5] \times [0.5, 1.5]$.

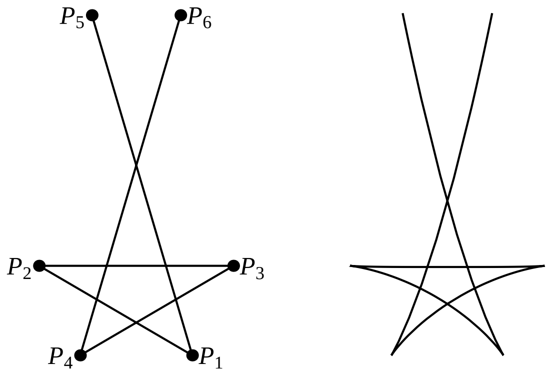

Figure 1. Left: The graph of the offset to the zero set of $f(x, y) = y - x^4$ at distance -1. Right: The offset curve at distance -1 to the curve defined by $f(x, y) = y - x^4$

Step 1: The support function of f is given by $27n_1^4 + 256h^3n_2 = 0$. The support function h_d of the offset o_1 to the zero set of f at distance -1 satisfies

$$27n_1^4 + 256(h_d + 1)^3 n_2 = 0.$$

Step 2: Solving the equation (5) we get four points corresponding to values $n_{1i},\ 1 \le i \le 4$ in the table bellow. There are no extremal points and two boundary points corresponding to values $n_{15},\ n_{16}$ in the lower table.

Step 3: Since $n_2 < 0$ for every point on $\mathcal{C}$, we can connect points using only n_1. The order is following: $P_5,\ P_1,\ P_2,\ P_3,\ P_4,\ P_6$. See figure 1.

Step 4: The coordinates of points are following:

$P_1:$	$n_{11} = -0.8430$	$[\ 0.1113, 0.8246]$
$P_2:$	$n_{12} = -0.0979$	$[-0.1930, 1.0023]$
$P_3:$	$n_{13} = 0.0979$	$[\ 0.1930, 1.0023]$
$P_4:$	$n_{14} = 0.8430$	$[-0.1113, 0.8246]$
$P_5:$	$n_{15} = -0.9795$	$[-0.0880, 1.5000]$
$P_6:$	$n_{16} = 0.9795$	$[\ 0.0880, 1.5000]$

Conclusion

We summarized the main ideas of existing algorithms for the determination of the topology of offset curves. Then we suggested a new approach to this problem using the support function representation of curves. In this representation we described cusps, extremal points and boundary points. We proposed an algorithm using these critical points to approximate the original curve by linear segments. Finally we applied our algorithm to convolutions and offsets, which we demonstrated on an example.

In the future, we intend to enlarge the algorithm by a Step 5, in which we find the exact position of self-intersections. Due to this step the resulting graph will be topologically equivalent to the input curve.

We also plan to improve the algorithm to cover all curves including the curves with multivaluated support function (i.e. with inflection points). All our results (e.g. computation of cusps coordinates) could be also used in the case when the curve has inflection points or when the support function is defined implicitly. Also the determination of inflection points is straightforward in the dual representation. The only thing which is not clear, and on which we are still working, is the connectivity of critical, resp. inflection points.

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