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Application of Peano Kernel

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Abstract. In the present work we have studied an error representation for a quadrature formula of Peano kernel and the practical application of this theory while looking for the optimal quadrature formula of Nikolskij's and Sard's types. We follow works [Ghizzeti and Ossicini, 1970] and [Sard, 1963]. First we describe construction of Peano kernel of a quadrature formula and its generalization and then we define the optimal quadrature formula of Nikolskij's and Sard's types. Further Peano kernel and its generalization are used for finding the optimal quadrature formula of Nikolskij's type. In the last section numerical results for one example are presented. The paper is a continuation of the Diploma thesis [Valešová, 2009].

Introduction

Quadrature formulae are methods for the approximate evaluation of definite integrals. They are used for computation of those integrals for which the integrand is available only at discrete points (for example from experimental measurement) or which we are not able to compute exactly. In the present work we consider a quadrature formula of two types:

$$Q_m f = \sum_{i=1}^{m} a_i f(x_i) \tag{1}$$

and

$$Q_m f = \sum_{l=0}^{n-1} \sum_{i=1}^{m} a_{li} f^{(l)}(x_i) \tag{2}$$

where x_i are called nodes of the quadrature formula Q_m and a_i or a_{li} are called coefficients of the quadrature formula, for $i = 1, \ldots, m$ and $l = 0, \ldots, n-1$, where $m, n \in \mathbb{N}$.
The theory of Peano kernel and its generalization can be found in [Engels, 1980] and [Ghizzeti and Ossicini, 1970]. Application of Peano kernel and its generalization for error term of a quadrature formula is used for construction of an error representation, in consequence for construction of an error estimation.

Peano kernel of a quadrature formula

First we consider a quadrature formula (1), in which are used only values of the integrand in nodes of the quadrature formula, not values of the derivatives of the integrated function f. Peano kernel of the quadrature formula can be defined in the following way:

Definition 1 *Let $f \in \mathcal{C}^{k+1}([0,1])$, let Q_m be a quadrature formula (1) of degree $k \geq 0$, let the error-functional E_m be defined by formula $E_m f = \int\limits_0^1 f(t)dt - Q_m f$. Then the function $K_{m,k}(t) \equiv E_m(x-t)_+^k$ is called the Peano kernel of the quadrature formula Q_m of degree k.*

There it follows very important theorem, which gives the error representation for a quadrature formula of the Peano kernel. The proof can be found in [Engels, 1980].

Theorem 1 *Let $f \in \mathcal{C}^{k+1}([0,1])$, where $k \geq 0$ and Q_m be a quadrature formula of degree $n \geq k$. Then the error-functional E_m can be represent as*

$$E_m f = \frac{1}{k!} \int\limits_0^1 K_{m,k}(t) f^{(k+1)}(t)dt \tag{3}$$

where $K_{m,k}(t)$ is the Peanovo kernel of the quadrature Q_m of degree k.
The error representation (3) can be used for construction of the error estimation

$$|E_m f| \leq \frac{1}{k!}\left(\int_0^1 K_{m,k}(t)^2\, dt\right)^{\frac{1}{2}}\left(\int_0^1 f^{(k+1)}(t)^2\, dt\right)^{\frac{1}{2}}, \tag{4}$$

which is very important, because properties of the quadrature formula Q_m and the integrated function f are separated in this estimation.

Generalized Peano kernel

In this section, it is necessary to define Peano kernel of a quadrature formula (2), which contains not only values of an integrated function f in nodes of a quadrature formula but also values of derivatives of the function f. Also integrals with nontrivial weight functions need to be approximated. In addition, it is very important to obtain an error representation of the quadrature formula (2) in the same form as (3). At the first degree of a quadrature formula need to be redefined. A linear differential operator D will be used for definition of the new degree of a quadrature formula.
Definition 2 *The quadrature formula Q_m has degree $k \in \mathcal{N}$ if and only if the error-functional E_m, where $E_m f = \int_0^1 f(x)w(x)\, dx - Q_m f$, vanishes an arbitrary solution g of differential equation $Dg = 0$, where the linear differential operator D has degree k.*
We consider the differential operator $D = \frac{d^k}{dx^k}$. There it follows the definition of generalized Peano kernel.
Definition 3 *Let $f \in \mathcal{C}^k([0,1])$, let Q_m be the quadrature formula (2) of degree $k \in \mathcal{N}$, let E_m be the error-functional. If there exists a function $\mathbb{K}_{m,k-1}$ such that*

$$E_m f = \int_0^1 \frac{d^k f}{dt^k}(t)\mathbb{K}_{m,k-1}(t)\, dt \tag{5}$$

then the function $\mathbb{K}_{m,k-1}$ is called the generalized Peano kernel of the quadrature formula Q_m of degree $k-1$.
Definition 3 meets our requirements, because the quadrature formula (2) is considered and the error representation (5) is an analogy of the error representation (3).
Construction of generalized Peano kernel is based on Green-Lagrange identity (the proof can be found in [Ghizzeti and Ossicini, 1970]). The detailed description of the construction can be found in [Valešová, 2009].
Theorem 2. *Let $u, v \in \mathcal{C}^k([0,1]),\ k = 1, 2, \ldots$. Then*

$$v(x)u^{(k)}(x) - (-1)^k v^{(k)}(x)u(x) = \frac{d}{dx}\sum_{i=0}^{k-1}(-1)^{k-i-1}u^{(i)}(x)v^{(k-i-1)}(x)$$

for every $x \in [0,1]$.
We apply Green-Lagrange identity

$$v(x)D(f)(x) - D^*(v)(x)f(x) = \frac{d}{dx}\sum_{l=0}^{n-1} f^{(l)}(x)D^*_{n-l-1}(v)(x),$$

where the operators D^* or D^*_i are adjoint to D or D_i, where $D_i = \frac{d^i}{dx^i}$, for $i = 0, \ldots, n-1$. Then the linear differential equation

$$D^*(v) = w \tag{6}$$

can be constructed, where the function w is the weight function of the integral which is approximated. We are looking for a function $\varphi_i(x)$, which solves the differential equation (6) on the interval $[x_i, x_{i+1})$, for $i = 0, \ldots, m$. We obtain equation

$$\varphi_i(x)D(f)(x) - w(x)f(x) = \frac{d}{dx}\sum_{l=0}^{n-1} f^{(l)}(x)D^*_{n-l-1}(\varphi_i)(x), \tag{7}$$

for $i = 0 \ldots m$. Equations (7) can be integrated from x_i to x_{i+1}, for $i = 0 \ldots m$. The following equation is the sum of these integrals:

$$\int\limits_0^1 w(x)f(x)\;dx = -\sum_{i=0}^{m}\left[\sum_{l=0}^{n-1} f^{(l)}(x)D^*_{n-l-1}(\varphi_i)(x)\right]_{x_i}^{x_{i+1}} +$$

$$+\sum_{i=0}^{m}\int\limits_{x_i}^{x_{i+1}} \varphi_i(x)D(f)(x)\;dx$$

The function $\mathbb{K}_{m+2,k-1}$ can be defined by

$$\mathbb{K}_{m+2,k-1}(x)|_{[x_i,x_{i+1})} = \varphi_i(x) \tag{8}$$

for $i = 0, \ldots, m$. The function $\mathbb{K}_{m+2,k-1}$ meet requirements of generalized Peano kernel listed in the definition 3, therefore the function $\mathbb{K}_{m+2,k-1}$ can be called the generalized Peano kernel. Then the quadrature formula corresponding with the generalized Peano kernel $\mathbb{K}_{m+2,k-1}$ can be constructed:

$$\begin{aligned} Q_{m+2}f \;\; = \;\; & \sum_{l=0}^{n-1} f^{(l)}(x_0)D^*_{n-l-1}(\varphi_0)(x_0) + \\ & + \sum_{l=0}^{n-1}\sum_{i=1}^{m} f^{(l)}(x_i)\left(D^*_{n-l-1}(\varphi_i - \varphi_{i-1})(x_i)\right) - \\ & - \sum_{l=0}^{n-1} f^{(l)}(x_{m+1})D^*_{n-l-1}(\varphi_m)(x_{m+1}). \end{aligned}$$

The coefficients of the quadrature formula can be easily calculated by formulae

$$\begin{aligned} a_{l0} &= D^*_{n-l-1}(\varphi_0)(x_0) \\ a_{li} &= D^*_{n-l-1}(\varphi_i - \varphi_{i-1})(x_i), \quad i = 1, \cdots, m \\ a_{l(m+1)} &= -D^*_{n-l-1}(\varphi_m)(x_{m+1}). \end{aligned}$$

We obtain the quadrature formula corresponding to the generalized Peano kernel $\mathbb{K}_{m+2,k-1}$:

$$Q_{m+2}f = \sum_{l=0}^{n-1}\sum_{i=0}^{m+1} a_{li}f^{(l)}(x_i)$$

Remark Nodes x_0 and x_{m+1} are added, in case nodes of the quadrature formula $x_1, \ldots, x_m$ do not contain 0 and 1. For detailed description of the construction of generalized Peano kernel and further examples, please, see [Valešová, 2009].

Optimal quadrature formula

In this section, an idea of optimal quadrature formulae of Nikolskij's and Sard's types will be introduced. The error or the error estimation of quadrature formulae need to be minimized on a set of functions. We can minimize the error of the quadrature formula in the sense of the coefficients of the quadrature formula. The acquired quadrature formula is called the optimal quadrature formula of Nikolskij's type.

Definition 4 *If the error of the quadrature formula Q_m is minimized on the set of functions $\mathcal{O}$ by the quadrature formula Q_m^N, for fixed nodes x_i, where $i = 1, \dots, m$, then the quadrature formula Q_m^N is called the optimal quadrature formula of Nikolskij's type.*

$$\mathcal{E}^N = \inf_{\substack{a_{li} \in \mathbb{R} \\ i=1,\dots,m \\ l=0,\dots,n-1}} \mathcal{E}$$

where $\mathcal{E}$ is the error of the quadrature formula Q_m on the set of functions $\mathcal{O}$.

The error of the quadrature formula can be minimized not only by the coefficients but also by the nodes of the quadrature formula. The acquired quadrature formula is called the optimal quadrature formula of Sard's type.

Definition 5 *If the error of the quadrature formula Q_m is minimized on the set of functions $\mathcal{O}$ by the quadrature formula Q_m^S, for free nodes x_i, where $i = 1, \dots, m$, then the quadrature formula Q_m^S is called the optimal quadrature formula of Sard's type.*

$$\mathcal{E}^S = \inf_{\substack{a_{li} \in \mathbb{R} \\ x_i \in [0,1] \\ i=1,\dots,m \\ l=0,\dots,n-1}} \mathcal{E}$$

where $\mathcal{E}$ is the error of the quadrature formula Q_m on the set of functions $\mathcal{O}$.

Finding the optimal quadrature formula of Sard's type is much more complicated, because of nonlinear dependence of the generalized Peano kenel on the nodes of the quadrature formula, in contrast to linear dependence of the generalized Peano kenel on the coefficients of the quadrature formula.

Example

We are looking for the optimal quadrature formula of Nikolskij's type Q_3^N, where

$$Q_3 f = \sum_{l=0}^{2} \left[\alpha_l f^{(l)}(0) + \beta_l f^{(l)}(\frac{1}{2}) + \gamma_l f^{(l)}(1) \right],$$

on the set of functions $\mathcal{O} = \mathcal{C}^3([0,1])$, for the weight function $w = 1$. The differential equation

$$-\frac{d^3 u(t)}{dt^3} = 1 \tag{9}$$

needs to be solved. The solution u of the differential equation (9) has the form

$$u(t) = -\frac{t^3}{6} + at^2 + bt + c,$$

where $a, b, c \in \mathbb{R}$. Then the $\mathcal{L}^2$ norm of the function u can be minimized:

$$\min_{a,b,c \in \mathbb{R}} \int\limits_{x_{i-1}}^{x_i} u^2(t)\ dt,$$

where $i = 1, 2$. Minimizers are denoted by φ_{i-1}, for $i = 1, 2$. We obtain functions φ_0 and φ_1 in the forms
$\varphi_0(t) = -\frac{t^3}{6} + \frac{t^2}{8} - \frac{t}{40} + \frac{1}{960}$,

$\varphi_1(t) = -\frac{t^3}{6} + \frac{3t^2}{8} - \frac{11t}{40} + \frac{21}{320}$.
According to the form (8), the generalized Peano kernel has the form

$$\mathbb{K}_{3,2}(t) = \begin{cases} -\frac{t^3}{6} + \frac{t^2}{8} - \frac{t}{40} + \frac{1}{960} & t \in [0, \frac{1}{2}), \\ \\ -\frac{t^3}{6} + \frac{3t^2}{8} - \frac{11t}{40} + \frac{21}{320} & t \in [\frac{1}{2}, 1]. \end{cases}$$

The optimal quadrature formula of Nikolskij's type Q_3^N corresponding to the generalized Peano kernel can be constructed:

$$Q_3^N f = \sum_{l=0}^{2} [f^{(l)}(x_0) D^*_{2-l}(\varphi_0)(x_0) + f^{(l)}(x_1) \left(D^*_{2-l}(\varphi_1 - \varphi_0)(x_1) \right) -$$

$$-f^{(l)}(x_3) D^*_{2-l-}(\varphi_1)(x_2)]$$

The solution of this problem is obtained - the optimal quadrature formula of Nikolskij's type is

$$Q_3^N f = \frac{1}{4} f(0) + \frac{1}{40} f'(0) + \frac{1}{960} f''(0) + \frac{1}{2} f(\frac{1}{2}) + \frac{1}{480} f''(\frac{1}{2}) +$$

$$+ \frac{1}{4} f(1) - \frac{1}{40} f'(1) + \frac{1}{960} f''(1)$$

and the error estimation can be found by application of Hölder's inequality for formula (5) in following form:

$$|E_3 f| \leq 3,94 \cdot 10^{-4} \left\| f^{(3)} \right\|_2 .$$

For detailed computation and further examples, please, see [Valešová, 2009].

Conclusion

We have used a generalized Peano kernel for finding the optimal quadrature formula of Nikolskij's type. The optimal quadrature formula of Sard's type is still the open problem. We have studied quadrature formulae which approximate integrals with a trivial weight function. The problem for nontrivial weight functions is still open for further considerations. Further there is the problem of generalization of these methods for cubature formulae on two-dimensional or three-dimensional domain.

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