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From the History of Continued Fractions

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Abstract. A short history and a brief introduction to the theory of continued fractions is presented.

1. What is a continued fraction?

The so-called simple continued fraction is an expression of the following form:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}}}}}}.$$

The above expansion was given by **Christiaan Huygens**[1].

A **generalized continued fraction** is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots + \cfrac{b_n}{a_n + \ddots}}},$$

where symbols a_i and b_i, for $i = 1,2,\ldots$, are real numbers. The a's are called *partial denominators* and b's *partial numerators*.

The next notation for generalized continued fraction, which we shall use in what follows, was proposed by **Alfred Israel Pringsheim**[2] in 1898[3]:

$$a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{|a_3} \ldots .$$

In 1813 **Carl Friedrich Gauss**[4] used following notation for generalized continued fractions:

$$a_0 + \underset{i=1}{\overset{\infty}{K}} \frac{b_i}{a_i}.$$

[1] **Christiaan Huygens** (1629 – 1695) – a Dutch mathematician, physicist and astronomer, who patented the first pendulum clock. In 1862, he built an automatic planetarium. To ensure the correct ratios of orbits periods of planets he used continued fractions, as it described in his posthumously published book *Descriptio automati planetarii*.

[2] **Alfred Israel Pringsheim** (1850 – 1941) – a German mathematician also known for his contribution in real and complex functions.

[3] *Über die Konvergenz unendlicher Kettenbrüche*. Sb. Münch. 28, 2898.

[4] **Carl Friedrich Gauss** (1777 – 1855) – a German mathematician who in 1801 published his famous work *Disquisitiones Arithmeticae;* he worked in a wide variety of fields in both mathematics and physics including number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. He gave the first proof of fundamental theorem of algebra.

Here the K stands for *Kettenbrüche*, the German word for continued fraction.

If every b_i, for $i = 1, 2, \ldots$, are equal 1:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

as mentioned above, is called an **infinite simple continued fraction** and it is usually written in the form of $[a_0; a_1, a_2, \ldots]$.

A continued fraction of the type

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}},$$

is called a **finite simple continued fraction** and it is also written in the form $[a_0; a_1, a_2, \ldots, a_n]$. The numbers a_i, $i = 0, 1, \ldots$, are called **elements** of the continued fraction.

The term *"continued fractions"* was first used by **John Wallis**[5] in 1653 in his book *Arithmetica infinitorum*.

The following theorem is of fundamental importance in the theory of continued fractions:

Every real number α can be uniquely represented by a simple continued fraction $[a_0; a_1, a_2, \ldots]$ where a_0 is an integer and a_i, $i = 1, 2, \ldots$, are positive integers. This continued fraction is finite if α is a rational number and infinite if α is an irrational number.

This theorem was proved by **Leonhard Euler**[6]. His first arithmetical paper on the subject *De fractionibus continuis* was published in 1737.

One way how to represent a rational number in the form of a continued fraction is given in the next example.

Example 1:

Express rational number $\frac{105}{38}$ as a continued fraction with integral elements.

Solution:

We use the Euclid´s algorithm for computation of the greatest common divisor of two natural numbers:

$$105 = 38 \cdot \underline{2} + 29$$
$$38 = 29 \cdot \underline{1} + 9$$
$$29 = 9 \cdot \underline{3} + 2$$
$$9 = 2 \cdot \underline{4} + 1$$
$$2 = 1 \cdot \underline{2} + 0$$

[5] **John Wallis** (1616 – 1703) – an English mathematician who built on Cavalieri's method of indivisibles to devise a method of interpolation.

[6] **Leonhard Euler** (1707 – 1783) – a Swiss mathematician, physicist and philosopher who made enormous contributions to a wide range of mathematics and physics including analytic geometry, trigonometry, geometry, calculus and number theory. His celebrated book *Introductio in analysin infinitorum*, published in Lausanne in 1748, contains the first extensive and systematic exposition of the theory of continued fractions.

This yields the following expansion of $\frac{105}{38}$ in the form of a continued fraction:

$$\frac{105}{38} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}} = [2;1,3,4,2].$$

To expand an irrational number in the form of a continued fraction we use another procedure, which when applied to a rational number coincides in principle with the above algorithm.

<u>Example 2:</u>

Expand the irrational number $\sqrt{2}$ into a continued fraction with integral elements.

<u>Solution:</u>

We use the function the integral part of a real number:

$a_0 = \lfloor \sqrt{2} \rfloor = 1$,

$r_1 = \frac{1}{\sqrt{2} - \lfloor \sqrt{2} \rfloor} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$, $\quad a_1 = \lfloor r_1 \rfloor = \lfloor \sqrt{2} + 1 \rfloor = 2$,

$r_2 = \frac{1}{r_1 - \lfloor r_1 \rfloor} = \frac{1}{\sqrt{2} + 1 - 2} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$, $\quad a_2 = \lfloor r_2 \rfloor = \lfloor \sqrt{2} + 1 \rfloor = 2$.

Continuing this process, we obtain $a_1 = a_2 = a_3 = \ldots = 2$ and we get:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = [1; 2, 2, 2, \ldots].$$

The resulting continued fraction for $\sqrt{2}$ is a *periodic continued fraction*: $\sqrt{2} = [1; \overline{2}]$.

It can be shown that quadratic polynomials with integral coefficients have periodic (simple) continued fraction expansion (for details consult [Pe], Chapter 3).

Finally we show an interesting continued fraction expansion for the **golden section**. Consider the quadratic equation $g^2 - g - 1 = 0$, which has two roots $g = \frac{1+\sqrt{5}}{2}$ and $g' = \frac{1-\sqrt{5}}{2}$. The positive root $g = \frac{1+\sqrt{5}}{2}$ is known as the golden section. From equation $g^2 - g - 1 = 0$ we obtain $g = 1 + \frac{1}{g}$. If we substitute g to $1 + \frac{1}{g}$ and we repeat this procedure we get the continued fraction for the golden section:

$$g = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}} = [1;1,1,1,\ldots].$$

2. Brief History of continued fractions

The history of continued fractions is long and it actually begins in a hidden form with approximation of quadratic irrationals, like $\sqrt{2}$, in ancient cultures. Another appearance of the expansion is it connection with one of the best known algorithms, the **Euclid**'s[7] algorithm as it was demonstrated in one of the above examples. Continued fractions can also be found in a hidden form in the work of **Eudoxos**[8], the author of *the exhaustive method*. One of the first explicit appearance of continued fractions can be found in the work of **Leonardo of Pisa**, known as **Fibonacci**[9]. In his book *Liber Abacci*, written in 1202, he introduced a kind of ascending continued fraction with the following meaning:

$$\frac{e}{f}\frac{c}{d}\frac{a}{b} = \frac{adf + cf + e}{bdf} = \frac{a}{b} + \frac{c}{d}\frac{1}{b} + \frac{e}{f}\frac{1}{b}\frac{1}{d}.$$

In modern mathematic symbolism we can rewrite this in the form:

$$\frac{a + \frac{c + \frac{e}{f}}{d}}{b}.$$

Fibonacci gives two other notations for continued fractions, one of them is described as follows:

"And if on the line there should be many fractions and the line itself terminated in a circle, then its fractions would denote other than what has been stated, as in this $\cdot\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}O$*, the line of which denotes the fractions eight-ninths of a unit, and six-sevenths of eight-ninths, and four-fifths of six-sevenths of eight-ninths, and two-thirds of four-fifths of six-sevenths of eight-ninths of an integral unit."*

This is the second notation:

$$\begin{array}{llll} & & & +\frac{384}{3} \\ & & +\frac{192}{5} & \\ & +\frac{48}{7} & & \\ \frac{8}{9} & & & \end{array}$$

If we rewrite the beginning of his text into mathematical symbolism we get:

$$\frac{8}{9} + \frac{6}{7}\cdot\frac{8}{9} + \frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9} = \frac{8}{9}\cdot\left(1 + \frac{6}{7}\cdot\left(1 + \frac{4}{5}\cdot\left(1 + \frac{2}{3}\right)\right)\right) = \frac{24}{9} = \frac{8}{3}.$$

The second notation is equal to continued fraction:

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[7] **Euclid of Alexandria** (c. 340 B. C. – c. 278 B. C.) – a Greek mathematician, his the most famous work is *The Elements*, which is consisting of 13 books.

[8] **Eudoxos** (c. 408 B. C. – 355 B. C.) – a Greek mathematician and astronomer, he mapped the stars and compiled a map of the known world.

[9] **Fibonacci** (c. 1170 – c. 1250) – a problem in the third section of *Liber abaci* led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which Fibonacci is best remembered today.

$$\frac{8+\cfrac{48+\cfrac{192+\frac{384}{3}}{5}}{7}}{9}=\frac{8+\frac{48+64}{7}}{9}=\frac{8+16}{9}=\frac{24}{9}=\frac{8}{3}.$$

The early history of ascending continued fractions begins with the so-called *Rhind papyrus*[10]. The Egyptians expressed fractions in terms of unit fractions $1/n$, $n>1$ a positive integer, and the fraction $2/3$. For example, the volume unit was called the *"hekat"*. As an example consider the problem to divide 5 hekat of wheat among 12 persons. We begin by partitioning each hekat into 3 parts and giving one of them to each person. Three parts remain. Each of these is divided into 4 parts, one of which is given to each person, that is $\frac{1}{4}$ of a $\frac{1}{3}$ of a hekat. Thus, each person has received $\frac{5}{12}=\frac{1}{3}+\frac{1}{12}$ of the total amount of wheat.

In other words

$$\frac{5}{12}=\frac{1}{3}+\frac{1}{4}\frac{1}{3}=\frac{1+\frac{1}{4}}{3},$$

which is an ascending continued fraction.

The real discovery of continued fractions is due to two mathematicians from the university Bologna, one of the first European universities. The first of these mathematicians is **Rafael Bombelli**[11]. In 1579 he published the second edition of his book *L´Agebra Opera.* In this book he gave an algorithm for extracting the square root of 13, which can be expressed using the modern notation in the form:

$$\sqrt{13}=3+\cfrac{4}{6+\cfrac{4}{6+\dots}}.$$

The second of these mathematicians was the Italian mathematician **Pietro Antonio Cataldi**[12], professor of mathematics in Florence and Bologna. Cataldi followed the same method as Bombelli for extracting the square roots, but he was the first to develop the symbolism for continued fractions and gave some of their properties. In 1606 Cataldi defined ascending continued fractions as *"a quantity written or proposed in the form of a fraction of a fraction."* Cataldi introduced the following first formal notation for the generalized continued fraction[13]

$$a_0\cdot\&\frac{n_1}{d_1.}\&\frac{n_2}{d_2.}\&\dots.$$

In 1682, Christiaan Huygens built, as mentioned, an automatic planetarium and he used continued fractions for finding the smallest integers whose ratio between the orbit period of the Earth and the orbit period of the Saturn is closest to the real one[14].

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[10] The **Rhind papyrus** is not a mathematical book in the modern sense. The Rhind papyrus is now in the British Museum in London and fractionated into two parts, which were bought by **Alexander Henry Rhind** (1833 – 1863) in Luxor in 1858. The papyrus was first translated and explained by **Wilhelm Eisenlohr** (1799 – 1872). It is dated in the 33rd year of the Hyksos king Aauserre Apophis, who must have ruled between 1788 and 1580 B. C.

[11] **Rafael Bombelli** (1526 – 1572) – he was an engineer and architect and he was the founder of imaginary numbers.

[12] **Pietro Antonio Cataldi** (1548 – 1626) – he was professor of mathematics and astronomy at the universities of Firenze, Perugia and Bologna. He wrote several treatises on arithmetic, number theory, perfect numbers and algebra.

[13] *Trattato del modo bravissimo di trovare la radice quadra delle numeri.* Bologna, 1613.

[14] *Opuscula posthuma. Descriptio automati planetarii.* Lugduni Batavorum, 1703.

The 18th century was a golden age of continued fractions. Three great mathematicians studied continued fractions:

Leonhard Euler[15] gave the first basic recurrence relations for the theory of continued fractions and he showed how to transform continued fractions into a series. He also derived a already used general method for the solution of the quadratic equation $x^2 = ax + b$, which can be written as $x = a + \frac{b}{x}$ and he thus obtained

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \ddots}}};$$

Johan Heinrich Lambert[16] studied how to express trigonometric functions in terms of continued fractions[17], for example:

$$\mathrm{tg}x = \cfrac{1}{\frac{1}{x} - \cfrac{1}{\frac{3}{x} - \cfrac{1}{\frac{5}{x} - \cfrac{1}{\frac{7}{x} - \ddots}}}};$$

Joseph Louis Lagrange[18] showed that the continued fractions expansion of $\sqrt{D}$ [19], where D is a non-square real positive number is periodic with an initial term a_0, and that the period has one of the following forms: $\left[a_0; a_1, \dots, a_n, a_n, \dots, a_1, 2a_0\right]$ or $\left[a_0; a_1, \dots, a_n, k, a_n, \dots, a_1, 2a_0\right]$, where k is a positive integer depending on D; in 1767 he published a method for approximating a real root of a polynomial equation by continued fractions; in 1776 he published a fundamental paper on the use of continued fractions in the integral calculus.

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[15] *Introductio in analysin infinitorum.* 2 vols. Lausanne, 1748

[16] **Johan Heinrich Lambert** (1728 – 1777) – he was the first to provide a rigorous proof that π is irrational. He was a colleague of Euler and Lagrange at the Academy of Sciences in Berlin.

[17] *Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques.* Historie de l´Académie, Berlin, 1761.

[18] **Joseph Louis Lagrange** (1736 – 1813) – in 1766, he gave the first proof that $x^2 = Dy^2 + 1$ has integral solutions with $y \neq 0$ if D is a given non square integer.

[19] *Additions au mémoire sur la résolution des équations numériques*. Mém. Berl. 24, année 1770 = Oeuvres, II.