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On the Complexity of Recognition of 2-collapsibility

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Abstract. A simplicial complex is d-collapsible if it can be reduced to an empty simplex by repeatedly removing face of dimension less than d that it is contained in a unique maximal face. These complexes have been studied since every complex that is the nerve of a family of convex sets in $\mathbb{R}^d$ is d-collapsible.

We show that the question whether a given complex is 2-collapsible can be solved in polynomial time.

Introduction

In 1975, Wegner [Wegner, 1975] introduced the notion of *d-collapsible* simplicial complexes (all the simplicial complexes throughout the text are assumed to be finite). He proved that d-collapsible complexes are *d-Leray* and also that every *d-representable* complex is d-collapsible. Precise definitions are given below.[1]

***d*-representable complexes.** For motivation, we introduce d-representable complexes before describing d-collapsible ones.

Famous Helly's theorem [Helly, 1923] asserts that if $C_1, C_2, \ldots, C_n$ are convex sets in $\mathbb{R}^d$, $n \geq d+1$ and every $d+1$ of them have a common point, then $C_1 \cup C_2 \cup \cdots \cup C_n \neq \emptyset$. This theorem (and several other theorems in discrete geometry) deals with *intersection patterns* of convex sets in $\mathbb{R}^d$. It can be restated using the notion of d-representable complexes, which "record" the intersection patterns.

The *nerve* of a family $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of sets is the simplicial complex with vertex set $[n] = \{1, 2, \ldots, n\}$ and with the set $\sigma \subseteq [n]$ forming a face if $\bigcap_{i\in\sigma} S_i \neq \emptyset$. A simplicial complex is d-representable if it is isomorphic to the nerve of a family of convex sets in $\mathbb{R}^d$.

In this language, Helly theorem states that if a d-representable complex contains all faces of dimension at most d, then it is already a *full simplex*. Beside Helly's theorem we also mention several other known results that can be formulated using d-representability. They include the fractional Helly theorem of Katchalski and Liu [Katchalski and Liu, 1979], the colorful Helly theorem of Lovász ([Lovász, 1974]; see also [Bárány, 1982]), the (p, q)-theorem of Alon and Kleitman [Alon and Kleitman, 1992], and the Helly type result of Amenta [Amenta, 1996].

***d*-collapsible complexes.** A face σ of a simplicial complex K is *collapsible* if it is contained in a unique inclusionwise maximal $\tau(\sigma) \in \mathsf{K}$ (we allow $\tau(\sigma) = \sigma$). Moreover, if $\dim \sigma \leq d-1$, then σ is *d-collapsible*. The *elementary d-collapse* of a d-collapsible face σ in K is the operation

$$\mathsf{K} \to \mathsf{K}_\sigma = \mathsf{K} \setminus [\sigma, \tau(\sigma)]$$

where $[\sigma, \tau(\sigma)] = \{\eta \in \mathsf{K} \mid \eta \supseteq \sigma\}$. A complex K is *d-collapsible* if there is a sequence of elementary d-collapses

$$\mathsf{K} \to \mathsf{K}_2 \to \mathsf{K}_3 \to \cdots \to \emptyset.$$

[1] We assume that the reader is familiar with simplicial complexes (introductory chapters of books like [Matoušek, 2003; Hatcher, 2001] should provide a sufficient background).

We work with *abstract simplicial complexes*, i.e., set systems $\mathsf{K} \subseteq 2^V$ (on a given set of *vertices* V) such that if $A \in \mathsf{K}$ and $B \subseteq A$ then $B \in \mathsf{K}$. Elements of K are *faces* of K.

For several remarks on d-Leray complexes we also assume that the reader is familiar with homology. However, these remarks can be ignored—the remaining text is independent of them.

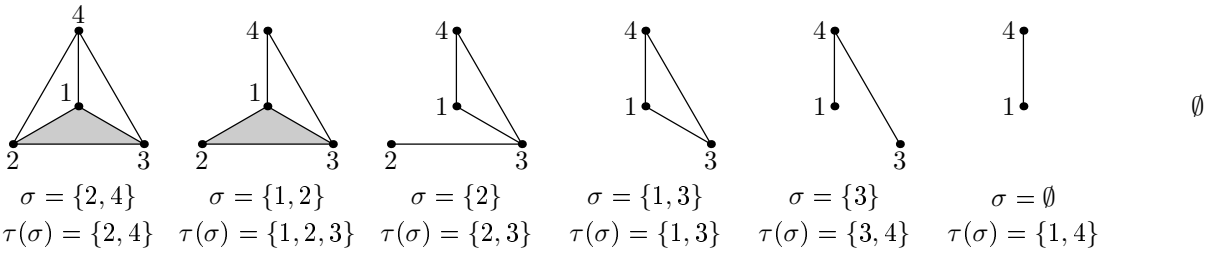


Figure 1. An example of 2-collapsing

An example of 2-collapsible complex is in Figure 1.

Many results on d-representable complexes can be generalized in terms of d-collapsible complexes (or even d-Leray complexes).[2] For example, a topological generalization of Helly's theorem follows from Helly's own work [Helly, 1930], a generalization of fractional Helly's theorem and (p,q)-theorem was done in [Alon et al., 2002], and a generalization of colorful Helly's theorem and Amenta's theorem was proved by Kalai and Meshulam [Kalai and Meshulam, 2005], [Kalai and Meshulam, 2008].

Dimensional gaps between collapsibility and representability were studied by Matoušek and this author [Matoušek and Tancer, 2008]; an interesting variation on d-collapsibility was used by Matoušek in order to show that it is not easy to remove degeneracy in LP-type problems [Matoušek, 2008].

The computational complexity of d-collapsibility. How hard is it to decide whether a given simplicial complex is d-collapsible? We consider the computational complexity (the size of an input is measured in the number of faces of the complex in question) of this question, regarding d as a fixed integer; we refer to it as d-COLLAPSIBILITY.

By Lekkerkerker and Boland [Lekkerkerker and Boland, 1962] (see also [Wegner, 1975]), 1-collapsible complexes are exactly 1-Leray complexes which coincide with *clique complexes* of *chordal graphs*. Hence 1-COLLAPSIBILITY is polynomial.

The main result of this paper is the following:

Theorem 1. *2-COLLAPSIBILITY is polynomial.*

On the other hand in [Tancer, 2008] it is shown, that d-COLLAPSIBILITY is NP-complete for $d \geq 4$.

Suppose that d is fixed and K is a simplicial complex. A *good face* is a d-collapsible face of K such that K_σ is d-collapsible; a *bad face* is d-collapsible face of K such that K_σ is not d-collapsible.

Now suppose that K is a d-collapsible complex. It is not immediately clear whether we can choose greedily elementary d-collapses in any order to d-collapse K, or there is a "bad sequence" of d-collapses such that the resulting complex is not d-collapsible anymore. Therefore, we consider the following question: For which d there is a d-collapsible complex K such that it contains a bad face? We show that for $d = 2$ there is no such a complex.

Theorem 2. *Let $d \leq 2$. Then every d-collapsible face of a d-collapsible complex is good.*

Theorem 1 is a straightforward consequence of Theorem 2, since we can greedily 2-collapse a given complex; this complex is 2-collapsible if and only if a greedy 2-collapsing yields an empty complex.

[2] A simplicial complex is d-Leray if its every induced subcomplex has vanishing homology of dimensions d and more (over $\mathbb{Q}$).

Related complexity results. Similarly as d-COLLAPSIBILITY, we can also consider the computational complexity of d-REPRESENTABILITY and d-LERAY COMPLEX. In the following three paragraphs we offer a brief discussion on these problems. Precisely stated results with detailed proofs and another background are in [Tancer, 2008].

By a modification of result of Kratochvíl and Matoušek on string graphs [Kratochvíl and Matoušek, 1989] (see also [Kratochvíl, 1991]), one has that 2-REPRESENTABILITY is NP-hard. Considering the $(d-2)$-tuple *suspension*[3] of complexes in the NP-reduction, we have that d-REPRESENTABILITY is NP-hard for $d \geq 2$.

In fact, an analogy of Theorem 2 for $d \geq 3$ is not true, since for every $d \geq 3$ there exists a d-collapsible complex with a bad d-collapsible face. This construction is also an intermediate step for showing that d-COLLAPSIBILITY is NP-complete for $d \geq 4$.

Finally, d-LERAY COMPLEX is polynomial. Indeed an equivalent characterization of a d-Leray complex K is that it is sufficient to test whether the homology (of dimension greater or equal to d) of *links*[4] of faces of K vanishes. These tests can be performed in polynomial time (note that the k-th homology of a complex of dimension less than k is always zero).

We see that the notions of d-representable, d-collapsible and d-Leray complexes have a different behavior in the complexity questions (although they behave similarly for Helly-type theorems as discussed above).

2-collapsibility

The contents of this section is the proof of Theorem 2.

If $d = 1$ then the result follows from the fact that 1-collapsible complexes are clique complexes of chordal graphs.

Now suppose that K is a 2-collapsible complex which, for contradiction, contains a bad 2-collapsible face $\sigma_B \in \mathsf{K}$. On the other hand, it also contains a good face σ_G since it is 2-collapsible. Moreover, suppose that K is the smallest complex (according to the number of faces) with these properties.

Claim 3. *Let σ be a good face of K and σ' be a 2-collapsible face of K_σ. Then σ' is a good face of K_σ.*

Proof. The complex K_σ is 2-collapsible since σ is a good face of K. If σ' was a bad face of K_σ then K_σ would be a smaller counterexample contradicting the choice of K. □

Recall that $\tau(\sigma)$ denotes the unique maximal superface of a collapsible face σ. Two collapsible faces σ and σ' are *independent* if $\tau(\sigma) \neq \tau(\sigma')$, otherwise they are *dependent*. The symbol $\mathrm{St}(\sigma, \mathsf{K})$ denotes the (open) *star* of simplex σ in K which consists of all superfaces of σ in K.

Claim 4. *Let $\sigma, \sigma' \in \mathsf{K}$ be independent 2-collapsible faces then σ is a 2-collapsible face of $\mathsf{K}_{\sigma'}$, σ' is a 2-collapsible face of K_σ, and $(\mathsf{K}_\sigma)_{\sigma'} = (\mathsf{K}_{\sigma'})_\sigma$.*

Proof. Since $\tau(\sigma) \neq \tau(\sigma')$ we have $\sigma \not\subseteq \tau(\sigma')$. Thus $\mathrm{St}(\sigma, \mathsf{K}) = \mathrm{St}(\sigma, \mathsf{K}_{\sigma'})$ implying that $\tau(\sigma)$ is also a unique maximal face containing σ when considered in $\mathsf{K}_{\sigma'}$. It means that σ is a collapsible face of $\mathsf{K}_{\sigma'}$. Similarly, σ' is a collapsible face of K_σ. Finally,

$$(\mathsf{K}_\sigma)_{\sigma'} = (\mathsf{K}_{\sigma'})_\sigma = \mathsf{K} \setminus \left\{ \eta \in \mathsf{K} \,\middle|\, \sigma \subseteq \eta \text{ or } \sigma' \subseteq \eta \right\}.$$

□

[3] Suppose that K and L are two simplicial complexes on disjoint vertex sets. The *join* of K and L is the complex $\mathsf{K} \star \mathsf{L} = \{\sigma \cup \eta \mid \sigma \in \mathsf{K}, \eta \in \mathsf{L}\}$. The suspension of a complex K is the complex $\mathsf{K} \star \mathsf{D}_2$, where D_2 is a discrete complex on 2 vertices (it contains two vertices and no other higher-dimensional faces).

[4] A link of a face σ in a complex K is the complex $\{\eta \in \mathsf{K} \mid \eta \cup \sigma \in \mathsf{K}, \eta \cap \sigma = \emptyset\}$.

Claim 5. *Any two 2-collapsible faces of* K *are dependent.*

Proof. For contradiction, let σ, σ' be two independent 2-collapsible faces in K. First, suppose that σ is good and σ' is bad (or vice versa). The face σ' is a collapsible face of K_σ by Claim 4. Thus, $(\mathsf{K}_\sigma)_{\sigma'}$ is 2-collapsible by Claim 3. But $(\mathsf{K}_\sigma)_{\sigma'} = (\mathsf{K}_{\sigma'})_\sigma$ by Claim 4, which contradicts the fact that σ' is a bad face.

Now suppose that σ and σ' are good faces. Then at least one of them is independent of σ_B, which yields the contradiction as in the previous case. Similarly, if both of σ and σ' are bad faces then at least one of them is independent of σ_G. □

Due to Claim 5 there exists a universal $\tau \in \mathsf{K}$ such that $\tau = \tau(\sigma)$ for every 2-collapsible $\sigma \in \mathsf{K}$. Let us remark that $\mathsf{K} \neq 2^\tau$ since σ_B is a bad face.

The following claim represents a key difference between 2-collapsibility and d-collapsibility for $d \geq 3$. It would not be valid in case of d-collapsibility.

Claim 6. *Let* σ *be a good face and* σ' *be a bad face. Then* $\sigma \cap \sigma' = \emptyset$.

Proof. It is easy to prove the claim in the case when either σ or σ' is a 0-face. Thus let us consider the case when both σ and σ' are 1-faces. For contradiction suppose that $\sigma \cap \sigma' \neq \emptyset$, i.e. $\sigma = \{u, v\}$, $\sigma' = \{v, w\}$ for some mutually different $u, v, w \in \tau$. Then $\tau \setminus \{u\}$ is the unique maximal face in K_σ that contains σ', so $(\mathsf{K}_\sigma)_{\sigma'}$ exists. Similarly $(\mathsf{K}_{\sigma'})_\sigma$ exists and the same argument as in the proof of Claim 4 yields $(\mathsf{K}_\sigma)_{\sigma'} = (\mathsf{K}_{\sigma'})_\sigma$. Similarly as in the proof of Claim 5, $(\mathsf{K}_\sigma)_{\sigma'}$ is 2-collapsible (due to Claim 3) but it contradicts the fact that σ' is a bad face. □

The complex K is 2-collapsible, let $\mathsf{K} = \mathsf{K}_1 \to \mathsf{K}_2 \to \cdots \to \mathsf{K}_m = \{\emptyset\}$ be a d-collapsing of K, where $\mathsf{K}_i \to \mathsf{K}_{i+1} = \mathsf{K}_i \setminus [\sigma_i, \tau_i]$ is an elementary d-collapse. Clearly, $\tau_1 = \tau$. Let k be a minimal integer such that $\tau_k \not\subseteq \tau$, such k exists since $\mathsf{K} \neq 2^\tau$.

Claim 7. *The face* σ_k *is a subset of* τ*, and it is not a 2-collapsible face of* K.

Proof. Suppose for contradiction that $\sigma_k \not\subseteq \tau$. Then $\mathrm{St}(\sigma_k, \mathsf{K}) = \mathrm{St}(\sigma_k, \mathsf{K}_i)$, since only subsets of τ are removed from K during the first i collapses. It implies that σ_k is a 2-collapsible face of K contradicting the definition of τ.

It is not a 2-collapsible face since it is contained in τ and $\tau_k \not\subseteq \tau$. □

From minimality of k, all the faces $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$ are good faces of K. Let $\eta = \sigma_k \cup \sigma_B$. See Figure 2. Claim 7 implies that $\eta \subseteq \tau$. By Claim 6 (and the fact that σ_k is not a good face—a consequence of Claim 7) the face η does not contain a good face. This implies that $\eta \in \mathsf{K}_k$. In particular $\eta \subseteq \tau_k$ since τ_k is the unique maximal face of K_k containing σ_k. We therefore have that $\sigma_B \subseteq \tau_k$. On the other hand τ is a unique maximal face of $\mathsf{K} \supseteq \mathsf{K}_k$ containing σ_B since σ_B is a 2-collapsible face. It is a contradiction to $\tau_k \not\subseteq \tau$. □

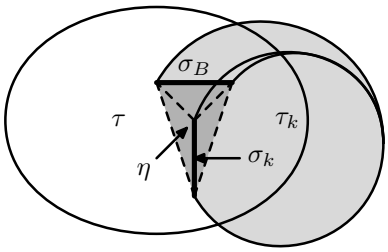


Figure 2. The simplices τ, τ_k and η.

Open problems

Since we already know that 2-collapsibility is polynomial time solvable and d-collapsibility is NP-complete for $d \geq 4$ the following question naturally arises. What is the computational complexity of 3-COLLAPSIBILITY?

In the context of d-representability, suspensions are a useful tool for passing among different values of d. Can they be used in the same way also for d-collapsibility? More precisely, is it true that a simplicial complex K is d-collapsible if and only if its suspension $\operatorname{susp}\mathsf{K}$ is $(d+1)$-collapsible?

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