UFY007
1. Diferenciální vektorové operátory

Hamiltonův operátor, nabla

$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ $\qquad \vec{i}.\vec{j} = \delta_{ij}$, $\qquad \vec{i}\times\vec{j} = \vec{k}$, $\qquad \vec{j}\times\vec{k} = \vec{i}$, $\qquad \vec{k}\times\vec{i} = \vec{j}$

gradient

$\vec{\nabla}\varphi = \text{grad}\ \varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$ $\qquad$ převádí skalární pole na vektorové pole

divergence

$\vec{\nabla}.\vec{A} = \text{div}\ \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ $\qquad$ převádí vektorové pole na pole skalární

rotace

$\vec{\nabla}\times\vec{A} = \text{rot}\ \vec{A} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$ $\qquad$ převádí vektorové pole na jiné vektorové pole

laplace

$\Delta = \vec{\nabla}.\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ $\qquad \Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$

$\Delta\vec{A} = \vec{i}.\Delta A_x + \vec{j}.\Delta A_y + \vec{k}.\Delta A_z$

Několik identit

$\text{div grad}\ s = \vec{\nabla}.\vec{\nabla}s = \Delta s$ $\qquad \text{rot grad}\ s = \vec{\nabla}\times\left(\vec{\nabla}s\right) = 0$

$\text{div rot}\ \vec{A} = \vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right) = 0$ $\qquad \text{rot rot}\ \vec{A} = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \text{grad div}\ \vec{A} - \Delta\vec{A}$

Geometrická definice operátorů

$\text{div}\ \vec{F} = \lim_{V_i\to 0}\frac{1}{V_i}\int_{S_i}\vec{F}.\vec{n}da_i$ $\qquad$ tok vektoru $\vec{F}$ plochou uzavírající elementární objem V_i

$\vec{n}.\text{rot}\,\vec{F} = \lim_{a_i \to 0} \frac{1}{a_i} \int_{C_i} \vec{F}.d\vec{s}$ cirkulace vektoru $\vec{F}$ křivkou omezující elementární plochu a_i

Gaussův – Ostrogradského teorém

$$\Phi_c = \int_S \vec{F}.\vec{n}da = \int_V \text{div}\,\vec{F}dV$$

Stokesův teorém

$$\Gamma = \int_c \vec{F}.d\vec{s} = \int_S \text{rot}\,\vec{F}.\vec{n}da$$

Křivočaré souřadnice

Válcové (cylindrické) souřadnice

$q_1 = \rho(0,\infty)$	$x = \rho \cos \varphi$	$\rho = \sqrt{x^2 + y^2}$
$q_2 = \varphi(0,2\pi)$	$y = \rho \sin \varphi$	$\varphi = \text{arctg}\frac{y}{x}$
$q_3 = z(-\infty,\infty)$	$z = z$	$z = z$

Kulové (sférické) souřadnice

$q_1 = r(0,\infty)$	$x = r \sin \vartheta \cos \varphi$	$r = \sqrt{x^2 + y^2 + z^2}$
$q_2 = \vartheta(0,\pi)$	$y = r \sin \vartheta \sin \varphi$	$\vartheta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}$
$q_3 = \varphi(0,2\pi)$	$z = r \cos \vartheta$	$\varphi = \text{arctg}\frac{y}{x}$

Lokální systémy souřadné – pravotočivý systém jednotkových vektorů

$\vec{e}_i.e_j = \delta_{ij}$ $\quad \vec{e}_1 \times \vec{e}_2 = \vec{e}_3,$ $\quad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1,$ $\quad \vec{e}_3 \times \vec{e}_1 = \vec{e}_2$

Derivace polohového vektoru $\vec{r}$ podle zobecněných souřadnic

Parciální derivace $\frac{\partial}{\partial q_i}$ má směr koordinační linie q_i, tedy jednotkového vektoru $\vec{e}_i$

$\frac{\partial \vec{r}}{\partial q_i} = H_i.\vec{e}_i$, H_i jsou Laméovy koeficienty

$$H_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$$

$\text{grad } q_i = h_i.\vec{e}_i$

$\text{grad } q_i . \frac{\partial \vec{r}}{\partial q_l} = \delta_{il}$ $\text{grad } q_i = \frac{\vec{e}_i}{H_i}$ (i = 1, 2, 3)

Určení koeficientů H_i

válcové souřadnice $H_\rho = 1$, $H_\varphi = \rho$, $H_z = 1$

kulové souřadnice $H_r = 1$, $H_\theta = r$, $H_\varphi = r \sin\theta$

Přírůstek polohového vektoru $d\vec{r}$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3 = H_1 dq_1 \vec{e}_1 + H_2 dq_2 \vec{e}_2 + H_3 dq_3 \vec{e}_3$$

$ds_i = H_i.dq_i$ jsou složky vektoru $d\vec{r}$,

$dV = ds_i.ds_2.ds_3 = H_1.H_2.H_3.dq_1.dq_2.dq_3$ je elementární objem v lokálním systému.

Vektorové operátory v křivočarých souřadnicích

$$\text{grad } \psi(q_1, q_2, q_3) = \frac{\partial \psi}{\partial q_1} \text{grad } q_1 + \frac{\partial \psi}{\partial q_2} \text{grad } q_2 + \frac{\partial \psi}{\partial q_3} \text{grad } q_3,$$

$$\text{grad } \psi = \frac{\vec{e}_1}{H_1} . \frac{\partial \psi}{\partial q_1} + \frac{\vec{e}_2}{H_2} . \frac{\partial \psi}{\partial q_2} + \frac{\vec{e}_3}{H_3} . \frac{\partial \psi}{\partial q_3}$$

$$\text{div } \vec{a} = \frac{1}{H_1 H_2 H_3} \left(\frac{\partial (a_1 H_2 H_3)}{\partial q_1} + \frac{\partial (a_2 H_1 H_3)}{\partial q_2} + \frac{\partial (a_3 H_1 H_2)}{\partial q_3} \right)$$

$(\text{rot } \vec{a})_1 = \frac{1}{H_2 H_3} \left(\frac{\partial (a_3 H_3)}{\partial q_2} - \frac{\partial (a_2 H_2)}{\partial q_3} \right)$, další složky cyklickou záměnou

$$\Delta \psi = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{H_1 H_2}{H_3} \frac{\partial \psi}{\partial q_3} \right) \right]$$

Válcové souřadnice

$$\text{grad } u = \vec{\nabla}.u = \frac{\partial u}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial u}{\partial \varphi} \vec{e}_\varphi + \frac{\partial u}{\partial z} \vec{e}_z$$

$$\text{div } \vec{A} = \vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$$

$$\text{rot } \vec{A} = \vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\vec{e}_\varphi + \left(\frac{\partial\left(\rho A_\varphi\right)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\vec{e}_z$$

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

Kulové souřadnice

$$\text{grad } u = \vec{\nabla}u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial\vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial u}{\partial\varphi}\vec{e}_\varphi$$

$$\text{div } \vec{A} = \vec{\nabla}\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial\left(\sin\vartheta\, A_\vartheta\right)}{\partial\vartheta} + + \frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\text{rot } \vec{A} = \vec{\nabla}x\vec{A} = \frac{1}{r\sin\vartheta}\left(\frac{\partial\left(\sin\vartheta\, A_\varphi\right)}{\partial\varphi} - \frac{\partial A_\vartheta}{\partial\varphi}\right)\vec{e}_r + \left(\frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial\left(r A_\varphi\right)}{\partial r}\right)\vec{e}_\vartheta + \frac{1}{r}\left(\frac{\partial\left(r A_\vartheta\right)}{\partial r} - \frac{\partial A_r}{\partial\vartheta}\right)\vec{e}_\varphi$$

$$\Delta u = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial^2 u}{\partial\varphi^2}$$

Několik příkladů použití diferenciálních operátorů

$$\vec{\nabla}\left(u.v\right) = u.\vec{\nabla}v + v\vec{\nabla}u \qquad \vec{\nabla}\left(u.\vec{A}\right) = u.\vec{\nabla}.\vec{A} + \vec{A}.\vec{\nabla}u$$

$$\left(\vec{A}.\vec{\nabla}\right)\vec{r} = \vec{A} \qquad \left(\vec{r} = \vec{e}_x.x + \vec{e}_y.y + \vec{e}_z.z\right)$$

$$\left(\vec{\nabla}.\vec{A}\right)\vec{B} = \left(\vec{A}.\vec{\nabla}\right).\vec{B} + \vec{B}\left(\vec{\nabla}.\vec{A}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{C}\times\vec{B}\right)\times\vec{A} = \vec{B}.\left(\vec{A}.\vec{C}\right) - \vec{C}.\left(\vec{A}.\vec{B}\right)$$

$$\vec{\nabla}.\left(\vec{A}\times\vec{B}\right) = \vec{B}.\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}\times\left(u.\vec{A}\right) = u.\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\times\vec{\nabla}u$$

$$\vec{\nabla}.\left(\vec{A}.\vec{B}\right) = \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{A}.\vec{\nabla}\right)\vec{B} + \left(\vec{B}.\vec{\nabla}\right)\vec{A}$$

$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}.\vec{\nabla}\right)\vec{A} - \left(\vec{A}.\vec{\nabla}\right)\vec{B} + \vec{A}.\left(\vec{\nabla}.\vec{B}\right) - \vec{B}\left(\vec{\nabla}.\vec{A}\right)$$