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General LORENTZ Transformation of DIRAC'S Wave Function

Dedicated to the Czech Academy of Sciences and Arts in honour of the memory of the late Prof. F. Záviška on the occasion of the 70th anniversary of his birth by

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(Presented November 18, 1949.)

During his stay at the Institute for Advanced Studies in Dublin Dr BRDIČKA[1]) has shown in which way the four components $\psi_1, \psi_2, \psi_3\, \psi_4$ of the DIRAC'S wave function ψ are transformed when we pass from the original coordinates $x_1, x_2, x_3, x_4 = ict$ to the new ones x_1', x_2', x_3', x_4' by means of the proper general LORENTZ transformation $x_k = a_{ik}x_i'$. He has given an explicit expression for the transformation matrix Λ, which brings the transformed function ψ' into relation with the original function ψ as expressed by $\psi' = \Lambda\psi$, provided that DIRAC'S matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$, which fulfil the relations $\gamma^i\gamma^k + \gamma^k\gamma^i = 2\delta^{ik}$, have an entirely special form[2]).

This problem has been considered very complicated[3]); in the literature, as far as I know, we do not find an explicit expression of Λ with the help of the coefficients a_{ik} of the general LORENTZ transformation[4]).

[1]) M. BRDIČKA, A Remark on Proper Lorentz Transformation of Dirac's Equations. Bulletin international de l'Académie tchèque des sciences et des arts, LI (1950).

[2]) See p. 19 of this paper.

[3]) W. PAULI, Contributions mathématiques à la théorie Dirac, Annales de l'Institut Henri Poincaré, VI, p. 124, 14th line from above.

[4]) P. A. M. DIRAC, Applications of Quaternions to Lorentz Transformations, Proceedings of the Royal Irish Academy, Dublin, L, sect. A, No 16, p. 261 (1945).

The endeavour to furnish a systematic solution of this problem irrespective of the choice of the matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ led me to the discovery of a relatively simple procedure how to find explicitely the wanted matrix Λ. It has come to light that there exists an infinite number of mutually equivalent solutions, some of which are conspicuous by their relative simplicity; one of them is the very solution given by BRDIČKA.

1. *The problem.* A linear orthogonal transformation

$$x_k = a_{ik}x_i', \quad x_i' = a_{ik}x_k, \quad i, k = 1, 2, 3, 4, \tag{1}$$

is called general LORENTZ transformation in case that x_1, x_2, x_3 are Cartesian coordinates for space and $x_4 = \sqrt{-1}\, ct$ is an imaginary coordinate for time, so that the coefficients a_{ik} with only one index which equals 4 are purely imaginary whilst all others are real.

This transformation does not change the form of DIRAC's equation

$$\{\gamma^k p_k + p_0\}\psi = 0, \quad p_0 = \mp i m_0 c, \tag{2}$$

where γ^k $(k = 1, 2, 3, 4)$ fulfil the relation $\gamma^k\gamma^l + \gamma^l\gamma^k = 2\delta^{kl}$. It merely changes into the equation

$$\{\gamma^i p_i' + p_0\}\psi' = 0, \tag{3}$$

where p_k and p_i' transform in exactly the same manner as x_k and x_i' according to the relations (1) and γ^i $(i = 1, 2, 3, 4)$ or γ^k $(k = 1, 2, 3, 4)$ do not transform; DIRAC's wave function ψ or ψ' transforms as follows:

$$\psi' = \Lambda\psi, \quad \psi = \Lambda^{-1}\psi', \tag{4}$$

where $\Lambda^{-1}\Lambda = 1$.

The problem is to find Λ as a function of the coefficients a_{ik} of LORENTZ's transformation and of DIRAC's matrices γ^k.

2. *The equation for determining* Λ. Since p_k transforms in the same way as x_k, the equation (2) changes at first into the form

$$\{\gamma^k a_{ik} p_i' + p_0\}\,\psi = 0. \tag{5}$$

Substituting for ψ' from the equation (4) into the equation (3) we obtain

$$\{\gamma^i p_i' + p_0\}\,\Lambda\psi = 0. \tag{6}$$

Carrying out the operation Λ^{-1} on this equation we arrive, in view of the relation $\Lambda^{-1}\Lambda = 1$, at

$$\{\Lambda^{-1}\gamma^i\Lambda p_i' + p_0\}\,\psi = 0. \tag{7}$$

The comparison of this equation with the equation (2) gives

$$\gamma^k p_k = \Lambda^{-1}\gamma^i\Lambda p_i'. \tag{8}$$

The same relation also applies to the coordinates x_k, x_i', so that the general LORENTZ transformation can also be written in the form[5])

[5]) Compare e. g. A. SOMMERFELD, Atombau u. Spektrallinien, II. Bd., p. 811, equation (29), (1939).

$$\gamma^k x_k = \Lambda^{-1}\gamma^i x_i'\Lambda, \quad \gamma^i x_i' = \Lambda\gamma^k x_k\Lambda^{-1}. \tag{9}$$

From an iteration of this it follows that $\sum_{i=1}^{4} x_i'^2 = \sum_{k=1}^{4} x_k^2$, which means that (1) is an orthogonal transformation. If we substitute for x_k and x_i' respectively in the left-hand sides of the equations (9) from equations (1), then a comparison of the coefficients of x_i' and x_k respectively yields[6])

$$a_{ik}\gamma^k = \Lambda^{-1}\gamma^i\Lambda = {}'\gamma^i, \quad a_{ik}\gamma^i = \Lambda\gamma^k\Lambda^{-1} = {}''\gamma^k, \tag{10}$$

which are two mutually equivalent relations, every one of which may be used for the determination of Λ. From equations (10) we arrive at the following, with due regard to interchangeability relations:

$$'\gamma^i\,'\gamma^j + {}'\gamma^j\,'\gamma^i = \Lambda^{-1}(\gamma^i\gamma^j + \gamma^j\gamma^i)\Lambda = 2\delta^{ij} = 2\sum_k a_{ik}a_{jk},$$

which are the well-known conditions for the orthogonal transformation. If Λ fulfils the relation

$$a_{ik}\gamma^k = \Lambda^{-1}\gamma^i\Lambda,$$

then the same relation is fulfilled by $L_1 = \pm\Lambda$, $L_1^{-1} = \pm\Lambda^{-1}$ and $L_2 = \pm i\Lambda$, $L_2^{-1} = \mp i\Lambda^{-1}$, but only one of these four possibilities contains an identical transformation of the function ψ.

3. *Hypercomplex number* Λ. In order to avoid adopting a special choice of the matrices for $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ we shall consider them as (CLIFFORD's) hypercomplex numbers. From these four basic hypercomplex numbers we obtain 16 elements of the group of hypercomplex numbers

$$\begin{gathered} 1 \\ \gamma^1, \gamma^2, \gamma^3, \gamma^4, \\ \gamma^{23}, \gamma^{31}, \gamma^{12}, \gamma^{14}, \gamma^{24}, \gamma^{34}, \\ \gamma^{234}, \gamma^{314}, \gamma^{124}, \gamma^{123}, \\ \gamma^{1234}, \end{gathered} \tag{11}$$

where γ^{ijkl} represents the product $\gamma^i\gamma^j\gamma^k\gamma^l$. Each hypercomplex number of this group may be expressed as a linear combination of these 16 elements with complex coefficients. Consequently even Λ can be expressed in this way, if we consider it as a hypercomplex number of this group. However we get the following important relation from (10):

$$'\gamma^{1234} = \Lambda^{-1}\gamma^{1234}\Lambda = \{\det.\,|a_{ik}|\}\,\gamma^{1234}. \tag{12}$$

Since the determinant of the orthogonal transformation is equal to ± 1, we have, for hypercomplex number Λ, the very important relation[7])

$$\Lambda\gamma^{1234} = \pm\,\gamma^{1234}\Lambda; \tag{13}$$

[6]) Compare e. g. W. PAULI, l. c., p. 124, equation (24) or A. SOMMERFELD, l. c., p. 258, equation (15).

[7]) Compare W. PAULI, l. c., p. 126, equation (28).

the $+$ sign holds good for the positive determinant (Lorentz's group proper), the $-$ sign for the negative determinant. In view of the anticommutability of hypercomplex numbers γ^1, γ^2, γ^3, γ^4, Λ can comprise only the elements written in the 1st, 3rd and 5th row of the tabulation (11) if $\{\text{det.}\ |a_{ik}|\} = 1$, while in the opposite case, i. e. if $\{\text{det.}\ |a_{ik}|\} = -1$, Λ can only comprise the elements written in the 2nd and 4th row.

Therefore, for $\{\text{det.}\ |a_{ik}|\} = +1$ we obtain

$$\Lambda^{(+)} = C_0 + C\gamma + C_{23}\gamma^{23} + C_{31}\gamma^{31} + C_{12}\gamma^{12} + C_{14}\gamma^{14} + C_{24}\gamma^{24} + C_{34}\gamma^{34}, \quad \gamma = -\gamma^{1234}, \tag{14}$$

while for $\{\text{det.}\ |a_{ik}|\} = -1$ we have

$$\Lambda^{(-)} = C_1\gamma^1 + C_2\gamma^2 + C_3\gamma^3 + C_4\gamma^4 + C_{234}\gamma^{234} + C_{314}\gamma^{314} + C_{124}\gamma^{124} + C_{123}\gamma^{123}. \tag{15}$$

Thereby the 16 elements of the group of hypercomplex numbers divide into two parts, each of which comprises 8 elements. The first part, which belongs to the determinant $+1$, is composed of the elements

$$1,\ \gamma,\ \gamma^{23},\ \gamma^{31},\ \gamma^{12},\ \gamma^{14},\ \gamma^{24},\ \gamma^{34}$$

and forms a sub-group of the original group. The second part, which belongs to the determinant -1, comprises the elements

$$\gamma^1,\ \gamma^2,\ \gamma^3,\ \gamma^4,\ \gamma^{234},\ \gamma^{314},\ \gamma^{124},\ \gamma^{123},$$

which, however, do not form a group.

If we multiply the elements of the first part with any element of the same part, the elements of this first part reproduce themselves, except for the sign and the order of the elements. If, on the other hand, we multiply the elements of the first part by any element of the second part, we obtain all elements of the second part except for the sign and the order of the elements. Therefore, if γ^A represents any element of the second part, we can write

either

$$\Lambda^{(-)} = \pm\, \Lambda^{(+)}\gamma^A, \quad \{\Lambda^{(-)}\}^{-1} = \pm\, \{\gamma^A\}^{-1}\{\Lambda^{(+)}\}^{-1}$$

or

$$\Lambda^{(-)} = \pm\, i\Lambda^{(+)}\gamma^A, \quad \{\Lambda^{(-)}\}^{-1} = \mp\, i\,\{\gamma^A\}^{-1}\{\Lambda^{(+)}\}^{-1},$$

so that

$$a_{ik}^{(-)}\gamma^k = \{\gamma^A\}^{-1}\{\Lambda^{(+)}\}^{-1}\gamma^i\Lambda^{(+)}\gamma^A.$$

Since, however,

$$a_{ik}^{(+)}\gamma^k = \{\Lambda^{(+)}\}^{-1}\gamma^i\Lambda^{(+)},$$

we obtain the relation

$$a_{ik}^{(-)}\gamma^k = a_{ik}^{(+)}\{\gamma^A\}^{-1}\gamma^k\gamma^A,$$

from which follow simple relations between the coefficients $a_{ik}^{(-)}$ and $a_{ik}^{(+)}$.

We could, however, with the same justification write either

$$\Lambda^{(-)} = \pm\, \gamma^A\Lambda^{(+)}, \quad \{\Lambda^{(-)}\}^{-1} = \pm\, \{\Lambda^{(+)}\}^{-1}\{\gamma^A\}^{-1}$$

or

$$\Lambda^{(-)} = \pm\, i\gamma^A\Lambda^{(+)}, \quad \{\Lambda^{(-)}\}^{-1} = \mp\, i\,\{\Lambda^{(+)}\}^{-1}\{\gamma^A\}^{-1},$$

so that

$$a_{ik}^{(-)}\gamma^k = \{\Lambda^{(+)}\}^{-1}\{\gamma^A\}^{-1}\gamma^i\gamma^A\Lambda^{(+)},$$

from which again follow simple relations between the coefficients $a_{ik}^{(-)}$ and $a_{ik}^{(+)}$.

From all this it is obvious that it will be sufficient to investigate the case in which the determinant of Lorentz's transformation is $+1$. And to this case we shall now turn our attention.

4. *The set of biquaternions Λ in the case of* $\{\text{det.}\ |a_{ik}|\} = 1$. The formula (14) can be written more symmetrically if we introduce the symbol $u^\pm = \frac{1}{2}(1 \pm \gamma)$, where $\gamma = -\gamma^{1234}$. Then

$$u^+ + u^- = 1, \quad (u^\pm)^2 = u^\pm, \qquad u^+ - u^- = \gamma, \quad u^\pm u^\mp = 0 \tag{16}$$

and Λ, Λ^{-1} respectively can be written as a sum of two sets of quaternions[8]) with units $u^\pm$ and three quaternions $\gamma^{23}\ u^\pm$, cycl.,

$$\begin{aligned} \Lambda &= (c_1\gamma^{23} + c_2\gamma^{31} + c_3\gamma^{12} + c_4)\,u^+ + (c_1'\gamma^{23} + c_2'\gamma^{31} + c_3'\gamma^{12} + c_4')u^-, \\ \Lambda^{-1} &= (c_1\gamma^{32} + c_2\gamma^{13} + c_3\gamma^{21} + c_4)u^+ + (c_1'\gamma^{32} + c_2'\gamma^{13} + c_3'\gamma^{21} + c_4')u^-. \end{aligned} \tag{17}$$

For it is possible to write the set of biquaternions Λ, in case that $\{\text{det.}\ |a_{ik}|\} = 1$, in the form

$$\begin{aligned} \Lambda = {} & (C_0 + C_{32}\gamma^{32} + C_{13}\gamma^{13} + C_{21}\gamma^{21})\,(u^+ + u^-) + \\ & + (C + C_{41}\gamma^{32} + C_{42}\gamma^{13} + C_{43}\gamma^{21})\,(u^+ - u^-), \end{aligned} \tag{18}$$

because $C_{ik} = -C_{ki}$,

$$\begin{aligned} & \gamma^{21}u^\pm = u^\pm\gamma^{21}, \ \text{cycl.}; \ \gamma^{43} = \gamma^{21}(u^+ - u^-) = (u^+ - u^-)\gamma^{21}, \ \text{cycl.} \\ & u^\pm\gamma^i = \gamma^i u^\mp, \ i = 1, 2, 3, 4, \ \gamma^i\gamma^{1234} = -\gamma^{1234}\gamma^i. \end{aligned} \tag{19}$$

The relation between the coefficients C_{ij}, C_0, C and the coefficients c_k is the following:

$$\left.\begin{aligned} C_{23} &= \tfrac{1}{2}(c_1 + c_1'), \\ C_{14} &= \tfrac{1}{2}(c_1 - c_1'), \end{aligned}\right\} \text{cycl.,} \quad \begin{aligned} C_0 &= \tfrac{1}{2}(c_4 + c_4'), \\ C &= \tfrac{1}{2}(c_4 - c_4'). \end{aligned} \tag{20}$$

It follows further from the equations (17), that

$$\Lambda^{-1}\Lambda = (c_1^2 + c_2^2 + c_3^2 + c_4^2)\,u^+ + (c_1'^2 + c_2'^2 + c_3'^2 + c_4'^2)\,u^- = 1,$$

i. e.

$$c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1 = c_1'^2 + c_2'^2 + c_2'^2 + c_4'^2. \tag{21}$$

We can rewrite the two relations (21), with help of the relations (20), in the following form:

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8) Compare W. Franz, Zur Methodik der Dirac-Gleichung, Sitzungsberichte der Akademie München, 1935, p. 390.

$$C_0^2 + C^2 + C_{23}^2 + C_{31}^2 + C_{12}^2 + C_{14}^2 + C_{24}^2 + C_{34}^2 = 1$$
$$C_0 C + C_{23} C_{14} + C_{31} C_{24} + C_{12} C_{34} = 0. \tag{22}$$

If, in addition, we introduce 8 further parameters by the relations

$$s_j = c_j + c_j', \quad s_j' = i(c_j - c_j'), \quad j = 1, 2, 3, 4, \tag{23}$$

both relations (21) pass into the form

$$\tfrac{1}{2}\sum_{j=1}^{4}(s_j^2 - s_j'^2) = 1, \quad \sum_{j=1}^{4} s_j s_j' = 0. \tag{24}$$

Finally, if we introduce 8 further parameters $\varkappa$, λ, μ, ν, $\varkappa'$, λ', μ', ν' by the following relations

$$\begin{aligned}
\varkappa &= \tfrac{1}{2}i(c_3 + c_3') + \tfrac{1}{2}(c_4 + c_4'), & \varkappa' &= \tfrac{1}{2}i(c_3 - c_3') + \tfrac{1}{2}(c_4 - c_4'),\\
\lambda &= \tfrac{1}{2}i(c_1 + c_1') - \tfrac{1}{2}(c_2 + c_2'), & \lambda' &= \tfrac{1}{2}i(c_1 - c_1') - \tfrac{1}{2}(c_2 - c_2'),\\
\mu &= \tfrac{1}{2}i(c_1 + c_1') + \tfrac{1}{2}(c_2 + c_2'), & \mu' &= \tfrac{1}{2}i(c_1 - c_1') + \tfrac{1}{2}(c_2 - c_2'),\\
\nu &= -\tfrac{1}{2}i(c_3 + c_3') + \tfrac{1}{2}(c_4 + c_4'), & \nu' &= -\tfrac{1}{2}i(c_3 - c_3') + \tfrac{1}{2}(c_4 - c_4'),
\end{aligned} \tag{25}$$

the two relations (21) will acquire the following form:

$$\begin{aligned}
\varkappa\nu - \lambda\mu + \varkappa'\nu' - \lambda'\mu' &= 1,\\
\varkappa\nu' + \varkappa'\nu - \lambda\mu' - \lambda'\mu &= 0.
\end{aligned} \tag{26}$$

The substitutions (20), (23), (25) are, however, only special cases of the general linear substitution

$$r_j = \beta_{j\varrho} c_\varrho, \quad c_k' = c_{4+k}, \quad k = 1, 2, 3, 4, \tag{27}$$

where j, ϱ pass through the values from 1 to 8 and the summation is carried out for those indices that appear twice; we assume the determinant D of this substitution to be different from zero, i. e.

$$D = \{\text{det.}\, |\beta_{j\varrho}|\} \neq 0.$$

Then the calculation of c_ν from the equations (27) gives

$$c_\nu = B_{j\nu} r_j, \tag{28}$$

where

$$B_{j\nu} = \frac{1}{D}\frac{\partial D}{\partial \beta_{j\nu}}. \tag{29}$$

I refrain from writing out the relations which will result from (21) by the substitution in it from (28).

If we put

$$r_j' = \beta_{j\varrho}' c_\varrho \tag{30}$$

and if we substitute (28) in this equation, we arrive at

$$r_j' = \beta_{j\varrho}' B_{k\varrho} r_k.$$

The relations (21) are, of course, the most symmetrical and the simplest. For this reason we shall first deal with the equations (17), (21) and (10). Since the following relations apply (compare with (16), (19)):

$$u^{\pm}\gamma^i u^{\pm} = u^{\pm}u^{\mp}\gamma^i = 0, \quad u^{\pm}\gamma^i u^{\mp} = u^{\pm}u^{\pm}\gamma^i = u^{\pm}\gamma^i = \gamma^i u^{\mp}, \tag{31}$$

we can simplify the determining equation (10) as follows:

$$\begin{aligned}
&\Lambda^{-1}\gamma^i\Lambda =\\
&= (c_1'\gamma^{32i} + c_2'\gamma^{13i} + c_3'\gamma^{21i} + c_4'\gamma^i)(c_1\gamma^{23} + c_2\gamma^{31} + c_3\gamma^{12} + c_4)u^+ +\\
&+ (c_1\gamma^{32i} + c_2\gamma^{13i} + c_3\gamma^{21i} + c_4\gamma^i)(c_1'\gamma^{23} + c_2'\gamma^{31} + c_3'\gamma^{12} + c_4')u^- =\\
&= a_{ik}\gamma^k.
\end{aligned} \tag{32}$$

5. *The preparation of the solution of the determining equation for Λ.* By comparison of the coefficients of γ^k on both sides of this equation we obtain, putting $c_{ik} = c_i c_k'$, the following equations:

$$\begin{aligned}
a_{11} &= c_{11} - c_{22} - c_{33} + c_{44}, & \{3\}\\
a_{44} &= c_{11} + c_{22} + c_{33} + c_{44}, & \{1\}\\
a_{41} &= c_{41} - c_{14} - c_{32} + c_{23}, & \{3\}\\
a_{14} &= -c_{41} + c_{14} - c_{32} + c_{23}, & \{3\}\\
a_{32} &= -c_{41} - c_{14} + c_{32} + c_{23}, & \{3\}\\
a_{23} &= c_{41} + c_{14} + c_{32} + c_{23}. & \{3\}
\end{aligned} \tag{33}$$

It is a system of 16 equations — the numbers in brackets indicate how many equations we get by cyclic variation of the indices 1, 2, 3 — for 16 unknowns c_{ik}, which may easily be determined. A development of this system (33) is the following system of 16 equations

$$\begin{aligned}
4c_{11} &= a_{11} - a_{22} - a_{33} + a_{44}, & \{3\}\\
4c_{44} &= a_{11} + a_{22} + a_{33} + a_{44}, & \{1\}\\
4c_{41} &= a_{41} - a_{14} - a_{32} + a_{23}, & \{3\}\\
4c_{14} &= -a_{41} + a_{14} - a_{32} + a_{23}, & \{3\}\\
4c_{32} &= -a_{41} - a_{14} + a_{32} + a_{23}, & \{3\}\\
4c_{23} &= a_{41} + a_{14} + a_{32} + a_{23}, & \{3\}
\end{aligned} \tag{34}$$

which is very similar to the system (33).

Using the relations (20) the system of 16 equations (33) takes the form

$$\begin{aligned}
a_{11} &= C_0^2 - C^2 + C_{23}^2 - C_{31}^2 - C_{12}^2 - C_{14}^2 + C_{24}^2 + C_{34}^2, & \{3\}\\
a_{44} &= C_0^2 - C^2 + C_{23}^2 + C_{31}^2 + C_{12}^2 - C_{14}^2 - C_{24}^2 - C_{34}^2, & \{1\}\\
a_{23} &= 2[\ \ (C_0 C_{23} - C C_{14}) + (C_{31}C_{12} - C_{24}C_{34})], & \{3\}\\
a_{32} &= 2[-(C_0 C_{23} - C C_{14}) + (C_{31}C_{12} - C_{24}C_{34})], & \{3\}\\
a_{14} &= 2[\ \ (C_0 C_{14} - C C_{23}) + (C_{24}C_{12} - C_{31}C_{34})], & \{3\}\\
a_{41} &= 2[-(C_0 C_{14} - C C_{23}) + (C_{24}C_{12} - C_{31}C_{34})]. & \{3\}
\end{aligned} \tag{35}$$

Similarly, by using the relations (25), we obtain from the equations (33) the following relations:

$$\begin{aligned}
a_{11} &= \tfrac{1}{2}(\varkappa^2 - \lambda^2 - \mu^2 + \nu^2 - \varkappa'^2 + \lambda'^2 + \mu'^2 - \nu'^2),\\
a_{21} &= \tfrac{1}{2}i(\varkappa^2 + \lambda^2 - \mu^2 - \nu^2 - \varkappa'^2 - \lambda'^2 + \mu'^2 + \nu'^2),\\
a_{31} &= -\varkappa\lambda + \mu\nu + \varkappa'\lambda' - \mu'\nu',\\
a_{41} &= i(\varkappa\lambda' + \mu'\nu - \varkappa'\lambda - \mu\nu'),
\end{aligned}$$

$$
\begin{aligned}
a_{12} &= \tfrac{1}{2} i(-\varkappa^2 + \lambda^2 - \mu^2 + \nu^2 + \varkappa'^2 - \lambda'^2 + \mu'^2 - \nu'^2),\\
a_{22} &= \tfrac{1}{2}(\varkappa^2 + \lambda^2 + \mu^2 + \nu^2 - \varkappa'^2 - \lambda'^2 - \mu'^2 - \nu'^2),\\
a_{32} &= i(\varkappa\lambda + \mu\nu - \varkappa'\lambda' - \mu'\nu'),\\
a_{42} &= \varkappa\lambda' + \mu\nu' - \varkappa'\lambda - \mu'\nu,\\
a_{13} &= -\varkappa\mu + \lambda\nu + \varkappa'\mu' - \lambda'\nu',\\
a_{23} &= i(-\varkappa\mu - \lambda\nu + \varkappa'\mu' + \lambda'\nu').\\
a_{33} &= \varkappa\nu + \lambda\mu - \varkappa'\nu' - \lambda'\mu'.\\
a_{43} &= i(\varkappa'\nu - \lambda'\mu - \varkappa\nu' + \lambda\mu'),\\
a_{14} &= i(-\varkappa\mu' + \lambda\nu' + \varkappa'\mu - \lambda'\nu),\\
a_{24} &= \varkappa\mu' + \lambda\nu' - \varkappa'\mu - \lambda'\nu,\\
a_{34} &= i(\varkappa\nu' + \lambda\mu' - \varkappa'\nu - \lambda'\mu),\\
a_{44} &= \varkappa\nu - \lambda\mu - \varkappa'\nu' + \lambda'\mu'.
\end{aligned}
\tag{36}
$$

I refrain from writing out the formulae for a_{ik} as functions of the parameters s_j, s_j'.

The equations (33) have a much clearer form than the equations (35) and (36) and they allow to derive easily the equations (34) which form the base for the determination of the coefficients c_k, c_k'; C, C_0, C_{ij}, s_j, s_j'; $\varkappa, \ldots, \nu'$.

Before starting with their determination let us bring to our mind the expression of the coefficients $c_1, \ldots, c_4'$ with the help of generalized EULER angles. For we may write:

$$
\begin{aligned}
c_1 &= \sin\tfrac{1}{2}\Theta \cos\tfrac{1}{2}(\omega - \varphi), & c_2 &= \sin\tfrac{1}{2}\Theta \sin\tfrac{1}{2}(\omega - \varphi),\\
c_3 &= \cos\tfrac{1}{2}\Theta \sin\tfrac{1}{2}(\omega + \varphi), & c_4 &= \cos\tfrac{1}{2}\Theta \cos\tfrac{1}{2}(\omega + \varphi),\\
c_1' &= \sin\tfrac{1}{2}\Theta' \cos\tfrac{1}{2}(\omega' - \varphi'), & c_2' &= \sin\tfrac{1}{2}\Theta' \sin\tfrac{1}{2}(\omega' - \varphi'),\\
c_3' &= \cos\tfrac{1}{2}\Theta' \sin\tfrac{1}{2}(\omega' + \varphi'), & c_4' &= \cos\tfrac{1}{2}\Theta' \cos\tfrac{1}{2}(\omega' + \varphi').
\end{aligned}
\tag{37}
$$

The equations (21) are then identically fulfilled. With the help of these expressions for $c_1, \ldots, c_4'$ we can find, by using the equations (33), the following formulae which express the transformation (1) by means of generalized EULER angles Θ, ω, φ, Θ', ω', φ':

$$
\begin{aligned}
a_{11} &= \cos\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta - \Theta') -\\
&\quad - \sin\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta + \Theta'),\\
a_{21} &= -\sin\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta - \Theta') -\\
&\quad - \cos\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta + \Theta'),\\
a_{31} &= -\sin\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \cos\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta + \Theta'),\\
a_{41} &= -\cos\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta - \Theta') -\\
&\quad - \sin\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta + \Theta').\\
a_{12} &= \cos\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \sin\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta + \Theta'),\\
a_{22} &= -\sin\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \cos\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega + \omega')\cos\tfrac{1}{2}(\Theta + \Theta'),
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
a_{32} &= -\sin\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta - \Theta') -\\
&\quad - \cos\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta + \Theta'),\\
a_{42} &= -\cos\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \sin\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega + \omega')\sin\tfrac{1}{2}(\Theta + \Theta').\\
a_{13} &= -\cos\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \sin\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta + \Theta'),\\
a_{23} &= \sin\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \cos\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta + \Theta'),\\
a_{33} &= -\sin\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \cos\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta + \Theta'),\\
a_{43} &= -\cos\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta - \Theta') -\\
&\quad - \sin\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta + \Theta').\\
a_{14} &= \cos\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \sin\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta + \Theta'),\\
a_{24} &= -\sin\tfrac{1}{2}(\varphi + \varphi')\cos\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \cos\tfrac{1}{2}(\varphi + \varphi')\sin\tfrac{1}{2}(\omega - \omega')\sin\tfrac{1}{2}(\Theta + \Theta'),\\
a_{34} &= \sin\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta - \Theta') +\\
&\quad + \cos\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta + \Theta'),\\
a_{44} &= \cos\tfrac{1}{2}(\varphi - \varphi')\cos\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta - \Theta') -\\
&\quad - \sin\tfrac{1}{2}(\varphi - \varphi')\sin\tfrac{1}{2}(\omega - \omega')\cos\tfrac{1}{2}(\Theta + \Theta').
\end{aligned}
\tag{38}
$$

However, let us return to the equations (34); they express 16 quantities $c_{ik} = c_i c_k'$ $(i,\ k = 1, 2, 3, 4)$ with the help of 16 coefficients a_{ik}. We have to determine from them 8 numbers $c_1, \ldots, c_4'$ which fulfil 2 relations (21), so that we have 18 equations available for 8 unknowns. But, out of the 16 coefficients a_{ik} only 6 are mutually independent, the remaining 10 are determined by 10 relations between the coefficients a_{ik} (the relations of orthogonality). Consequently, between 18 equations for 8 quantities $c_1, \ldots, c_4'$ 10 relations exist, so that we have 8 independent equations which are necessary and sufficient for the determination of 8 unknowns. It would not be difficult to express c_1', c_2', c_3', c_4' with the help of c_1 and then from the relation $\sum_{k=1}^{4} c_k'^2 = 1$ to determine c_1, so that c_1', c_2', c_3', c_4' will be wholly determined by the coefficients a_{ik}. Thereby the square root of a certain expression will be introduced into the denominators of the fractions expressing c_1', c_2', c_3', c_4'. By carrying through the same similarly with c_1, c_2, c_3, c_4 by means of c_1' and then finding the number c_1' from the equation $\sum_{k=1}^{4} c_k^2 = 1$ we shall determine c_1, c_2, c_3, c_4 wholly by the coefficients a_{ik}, but there will be a different square root in the denominators of the fractions expressing c_1, c_2, c_3, c_4 than before. For this reason this calculation of coefficients $c_1, \ldots, c_4'$ is not convenient. Therefore we choose another method which yields more suitable results. We shall, therefore, in addition, calculate the products

$$c_i c_k = \sum_{j=1}^{4} c_{ij} c_{kj}, \quad c_i' c_k' = \sum_{j=1}^{4} c_{ji} c_{jk}, \quad (i, k = 1, 2, 3, 4). \tag{39}$$

We shall easily recognise the validity of these relations, if we remember the relations (21).

However, before we turn to calculations, we shall deal with a special case from which will be, at least partly, apparent the subsequent procedure which we shall use later in the case of general Lorentz transformation with a determinant = 1.

6. *A special case* $a_{i4} = a_{4i} = 0$, $i = 1, 2, 3$. Let us choose the case of Lorentz's transformation with determinant $= +1$ which is characterized by six conditions, two of which read

$$\begin{aligned} 0 = a_{14} &= -c_{41} + c_{14} - c_{32} + c_{23}, \\ 0 = a_{41} &= \quad c_{41} - c_{14} - c_{32} + c_{23}, \end{aligned} \tag{40}$$

and the remaining four we obtain therefrom by cyclic variation of indices 1, 2, 3; the index 4 does not change.

The addition and the subtraction of these equations give

$$c_{32} = c_{23}, \quad c_{41} = c_{14}, \quad \text{cycl.}$$

or

$$c_3 c_2' = c_2 c_3', \quad c_4 c_1' = c_1 c_4', \quad \text{cycl.}; \tag{41}$$

these equations can only be fulfilled with ($\lambda \neq 0$)

$$c_k' = \lambda c_k, \quad k = 1, 2, 3, 4.$$

Then will $\sum_{k=1}^{4} c_k'^2 = \lambda^2 \sum_{k=1}^{4} c_k^2$, or with regard to (21)

$$\lambda^2 = 1; \tag{42}$$

consequently

$$c_k' = \pm c_k, \quad a_{44} = \sum_{k=1}^{4} c_k c_k' = \pm \sum_{k=1}^{4} c_k^2 = \pm 1. \tag{43}$$

The equations (34) will then be reduced; in case $a_{44} = 1$ it is necessary to choose the upper sign everywhere, in the case $a_{44} = -1$ the lower one:

$$\begin{aligned}
4c_1^2 &= \pm(a_{11} - a_{22} - a_{33}) + 1, & 4c_2c_1 &= \pm(a_{21} + a_{12}), \\
4c_1c_2 &= \pm(a_{21} + a_{12}), & 4c_2^2 &= \pm(-a_{11} + a_{22} - a_{33}) + 1, \\
4c_1c_3 &= \pm(a_{31} + a_{13}), & 4c_2c_3 &= \pm(a_{23} + a_{32}), \\
4c_1c_4 &= \pm(a_{23} - a_{32}), & 4c_2c_4 &= \pm(a_{31} - a_{13}), \\
4c_3c_1 &= \pm(a_{31} + a_{13}), & 4c_4c_1 &= \pm(a_{23} - a_{32}), \\
4c_3c_2 &= \pm(a_{23} + a_{32}), & 4c_4c_2 &= \pm(a_{31} - a_{13}), \\
4c_3^2 &= \pm(-a_{11} - a_{22} + a_{33}) + 1, & 4c_4c_3 &= \pm(a_{12} - a_{21}), \\
4c_3c_4 &= \pm(a_{12} - a_{21}), & 4c_4^2 &= \pm(a_{11} + a_{22} + a_{33}) + 1.
\end{aligned} \tag{44}$$

Now from the first column it follows that

$$\begin{aligned} c_1 &= k_{\mathrm{I}}\,[\pm(a_{11} - a_{22} - a_{33}) + 1], & c_2 &= \pm\, k_{\mathrm{I}}(a_{21} + a_{12}), \\ c_3 &= \pm\, k_{\mathrm{I}}\,(a_{31} + a_{13}), & c_4 &= \pm\, k_{\mathrm{I}}(a_{23} - a_{32}), \end{aligned} \tag{I}$$

where

$$(k_{\mathrm{I}})^{-2} = 4[\pm(a_{11} - a_{22} - a_{33}) + 1].$$

Similarly we get from the second column:

$$\begin{aligned} c_1 &= \pm\, k_{\mathrm{II}}(a_{21} + a_{12}), & c_2 &= k_{\mathrm{II}}[\pm(-a_{11} + a_{22} - a_{33}) + 1], \\ c_3 &= \pm\, k_{\mathrm{II}}(a_{23} + a_{32}), & c_4 &= \pm k_{\mathrm{II}}\,(a_{31} - a_{13}), \end{aligned} \tag{II}$$

where

$$(k_{\mathrm{II}})^{-2} = 4[\pm(-a_{11} + a_{22} - a_{33}) + 1].$$

The third column gives:

$$\begin{aligned} c_1 &= \pm\, k_{\mathrm{III}}(a_{31} + a_{13}), & c_2 &= \pm\, k_{\mathrm{III}}(a_{23} + a_{32}), \\ c_3 &= k_{\mathrm{III}}[\pm(-a_{11} - a_{22} + a_{33}) + 1], & c_4 &= \pm\, k_{\mathrm{III}}(a_{12} - a_{21}), \end{aligned} \tag{III}$$

where

$$(k_{\mathrm{III}})^{-2} = 4[\pm(-a_{11} - a_{22} + a_{33}) + 1].$$

And finally from the fourth column we obtain

$$\begin{aligned} c_1 &= \pm\, k_{\mathrm{IV}}(a_{23} - a_{32}), & c_2 &= \pm\, k_{\mathrm{IV}}(a_{31} - a_{13}), \\ c_3 &= \pm\, k_{\mathrm{IV}}(a_{12} - a_{21}), & c_4 &= k_{\mathrm{IV}}[\pm(a_{11} + a_{22} + a_{33}) + 1], \end{aligned} \tag{IV}$$

where

$$(k_{\mathrm{IV}})^{-2} = 4[\pm(a_{11} + a_{22} + a_{33}) + 1].$$

Since $(k_j)^{-1}$, $j = \mathrm{I, II, III, IV}$, has two values which differ in sign, we obtain from there a total of 2×4 possibilities of suitably expressing c_1, c_2, c_3, c_4 with the help of the elements of the determinant $|a_{ik}|$, $i, k = 1, 2, 3$, the value of which is $+1$ for $a_{44} = +1$ and the same number of possibilities for $a_{44} = -1$, when the value of that determinant is -1. We obtain, of course, an identical transformation of the wave function for identical transformation of coordinates in one case only, i. e. in case (IV), if we choose the upper sign and choose $(k_{\mathrm{IV}})^{-1}$ positive; the other formulae for this case fail to give results, because $(k_{\mathrm{I}})^{-2}$, $(k_{\mathrm{II}})^{-2}$, $(k_{\mathrm{III}})^{-2}$ for $a_{11} = a_{22} = a_{33} = 1$ are equal to zero. Since, however, $(k_j)^{-2}$ can never vanish at the same time for all four $j = \mathrm{I, II, III, IV}$, we always have the possibility of expressing c_1, c_2, c_3, c_4 with the help of the given coefficients of transformation a_{ik} ($i,k = 1, 2, 3$). In the case that $(k_j)^{-2} \neq 0$, $j = \mathrm{I, II, III, IV}$, the formulae (I), (II), (III), (IV) are mutually equivalent except for the sign. The possibility (IV) which, with a positive sign of the expression $(k_{\mathrm{IV}})^{-1}$ and with the upper sign gives an identical transformation of the wave function for identical transformation of coordinates, fails in the case of rotation through an angle π round the axis x, y, z respectively (in the first case $a_{11} = 1$, $a_{22} = -1$, $a_{33} = -1$, $a_{ik} = 0$, $i \neq k$; in the second case $a_{11} = -1$, $a_{22} = 1$, $a_{33} = -1$, $a_{ik} = 0$, $i \neq k$; in the third case $a_{11} = -1$, $a_{22} = -1$, $a_{33} = 1$, $a_{ik} = 0$, $i \neq k$), in which cases, however, the possibility (I), (II) or (III) is applicable respectively.

So far we have started from the set of quaternions

$$\Lambda = c_1\gamma^{23} + c_2\gamma^{31} + c_3\gamma^{12} + c_4.$$

We could have started equally well from the set of quaternions

$$L_1 = \gamma^{23}\Lambda \text{ or } L_2 = \gamma^{31}\Lambda \text{ or } L_3 = \gamma^{12}\Lambda.$$

Then the following will apply:

$$L_1^{-1} = \Lambda^{-1}\gamma^{32} \text{ or } L_2^{-1} = \Lambda^{-1}\gamma^{13} \text{ or } L_3^{-1} = \Lambda^{-1}\gamma^{21},$$

so that

$$L_j^{-1}L_j = 1, \; j = 1, 2, 3 \text{ (no summation)}$$

under the condition that

$$\sum_{k=1}^{4} c_k^2 = 1.$$

Using the transformation

$$x_k = \alpha_{ik}x_i', \; x_i' = \alpha_{ik}x_k,$$

we get

$$\alpha_{ik}\gamma^k = L_j^{-1}\gamma^i L_j,$$

so that

$$\begin{aligned}
&\alpha_{ik}\gamma^k = \Lambda^{-1}\gamma^{32i23}\Lambda = \pm a_{ik}\gamma^k, && \begin{matrix} + \text{ sign for } i = 1, \\ - \text{ sign for } i = 2{,}3 \end{matrix} \\
\text{or}& \\
&\alpha_{ik}\gamma^k = \Lambda^{-1}\gamma^{13i31}\Lambda = \pm a_{ik}\gamma^k, && \begin{matrix} + \text{ sign for } i = 2, \\ - \text{ sign for } i = 3{,}1 \end{matrix} \\
\text{or}& \\
&\alpha_{ik}\gamma^k = \Lambda^{-1}\gamma^{21i12}\Lambda = \pm a_{ik}\gamma^k, && \begin{matrix} + \text{ sign for } i = 3, \\ - \text{ sign for } i = 1{,}2. \end{matrix}
\end{aligned} \tag{45}$$

Therefore, from the formula

$$\Lambda = c_1\gamma^{23} + c_2\gamma^{31} + c_3\gamma^{12} + c_4, \; \Lambda^{-1} = c_1\gamma^{32} + c_2\gamma^{13} + c_3\gamma^{21} + c_4$$

it follows that

$$L_1 = \gamma^{23}\Lambda = \quad c_4\gamma^{23} + c_3\gamma^{31} - c_2\gamma^{12} - c_1, \; L_1^{-1} = \quad c_4\gamma^{32} + c_3\gamma^{13} - c_2\gamma^{21} - c_1$$

or

$$L_2 = \gamma^{31}\Lambda = -c_3\gamma^{23} + c_4\gamma^{31} + c_1\gamma^{12} - c_2, \; L_2^{-1} = -c_3\gamma^{32} + c_4\gamma^{13} + c_1\gamma^{21} - c_2$$

or

$$L_3 = \gamma^{12}\Lambda = \quad c_2\gamma^{23} - c_1\gamma^{31} + c_4\gamma^{12} - c_3, \; L_3^{-1} = \quad c_2\gamma^{32} - c_1\gamma^{13} + c_4\gamma^{21} - c_3.$$

With regard to (45) we thus have to carry out the following changes in the formulae for c_1, c_2, c_3, c_4:

Λ	c_1	c_2	c_3	c_4	a_{1k}	a_{2k}	a_{3k}	a_{4k}	(I)	(II)	(III)	(IV)
L_1	c_4	c_3	$-c_2$	$-c_1$	a_{1k}	$-a_{2k}$	$-a_{3k}$	a_{4k}	(IV)	(III)	—(II)	—(I)
L_2	$-c_3$	c_4	c_1	$-c_2$	$-a_{1k}$	a_{2k}	$-a_{3k}$	a_{4k}	—(III)	—(IV)	(I)	(II)
L_3	c_2	$-c_1$	c_4	$-c_3$	$-a_{1k}$	$-a_{2k}$	a_{3k}	a_{4k}	(II)	—(I)	(IV)	—(III)

Here (I), (II), (III), (IV) represent four possibilities for the group of four c_1, c_2, c_3, c_4, — (I) etc. means (I) with the opposite sign; $k = 1, 2, 3$. It is obvious, that the possibilities run practically into one another; we do not obtain essentially new possibilities (if we do not consider as a possibility the multiplication of the expression for Λ by the factor $\pm i$).

But, as we know from the paragraph 4, what we have just mentioned, is not the only choice for wanted numbers c_1, c_2, c_3, c_4.

For if we introduce for the case that $a_{44} = 1$ the four parameters $\varkappa$, λ, μ, ν, i. e. from the equation (25), by specialization of these equations (further four $\varkappa'$, λ', μ', ν' vanish), we get

$$\begin{aligned} \varkappa &= ic_3 + c_4, & \lambda &= \quad ic_1 - c_2, \\ \mu &= ic_1 + c_2, & \nu &= -ic_3 + c_4; \end{aligned} \tag{46}$$

the relations (26) then pass into a single relation

$$\varkappa\nu - \lambda\mu = 1, \tag{47}$$

from which we conclude $\varkappa$, λ, μ, ν to be well-known Cayley-Klein's parameters, because, by specialization (43) carried out in the equations (25) and (36) we arrive at the well known formulae

$$\begin{aligned}
&a_{11} = \tfrac{1}{2}(\varkappa^2 - \lambda^2 - \mu^2 + \nu^2), \; a_{12} = \tfrac{1}{2}i(-\varkappa^2 + \lambda^2 - \mu^2 + \nu^2), \\
&\qquad a_{13} = -\varkappa\mu + \lambda\nu, \\
&a_{21} = \tfrac{1}{2}i(\varkappa^2 + \lambda^2 - \mu^2 - \nu^2), \; a_{22} = \tfrac{1}{2}(\varkappa^2 + \lambda^2 + \mu^2 + \nu^2), \\
&\qquad a_{23} = -i(\varkappa\mu + \lambda\nu), \\
&a_{31} = -\varkappa\lambda + \mu\nu, \qquad a_{32} = i(\varkappa\lambda + \mu\nu), \qquad a_{33} = \varkappa\nu + \lambda\mu.
\end{aligned} \tag{48}$$

From these 9 equations and from the equation (47) 10 expressions $\varkappa^2$, λ^2, μ^2, ν^2, $\varkappa\lambda$, $\mu\nu$, $\varkappa\mu$, $\lambda\nu$, $\varkappa\nu$, $\lambda\mu$ can easily be calculated. But the same expressions can also be calculated from the equations (46) and (44). Therefore, if we calculate those 10 expressions from the equations (46) and (44) we can arrange them as follows:

$$\begin{aligned}
2\varkappa^2 &= \quad a_{11} + a_{22} + i(a_{12} - a_{21}), & 2\lambda\varkappa &= -a_{31} - ia_{32}, \\
2\varkappa\lambda &= -a_{31} - ia_{32}, & 2\lambda^2 &= -a_{11} + a_{22} - i(a_{12} + a_{21}), \\
2\varkappa\mu &= -a_{13} + ia_{23}, & 2\lambda\mu &= \quad a_{33} - 1, \\
2\varkappa\nu &= \quad a_{33} + 1, & 2\lambda\nu &= \quad a_{13} + ia_{23}, \\
2\mu\varkappa &= -a_{13} + ia_{23}, & 2\nu\varkappa &= \quad a_{33} + 1, \\
2\mu\lambda &= \quad a_{33} - 1, & 2\nu\lambda &= \quad a_{13} + ia_{23}, \\
2\mu^2 &= -a_{11} + a_{22} + i(a_{12} + a_{21}), & 2\nu\mu &= \quad a_{31} - ia_{32}, \\
2\mu\nu &= \quad a_{31} - ia_{32}, & 2\nu^2 &= \quad a_{11} + a_{22} - i(a_{12} - a_{21}).
\end{aligned} \tag{49}$$

From there we again have four equivalent possibilities of which I shall only write out the first one:

$$\varkappa = \frac{a_{11} + a_{22} + i(a_{12} - a_{21})}{\sqrt{2}\sqrt{a_{11} + a_{22} + i(a_{12} - a_{21})}},$$
$$\lambda = \frac{-a_{31} - ia_{32}}{\sqrt{2}\sqrt{a_{11} + a_{22} + i(a_{12} - a_{21})}}, \tag{50}$$
$$\mu = \frac{-a_{13} + ia_{23}}{\sqrt{2}\sqrt{a_{11} + a_{22} + i(a_{12} - a_{21})}},$$
$$\nu = \frac{a_{33} + 1}{\sqrt{2}\sqrt{a_{11} + a_{22} + i(a_{12} - a_{21})}}.$$

The square root in the denominator has a double sign; if we choose in the denominator the positive sign, we obtain also from there an identical transformation of the wave function for an identical transformation of coordinates.

However, by using (44) and (46) we get further four possibilities, which are equivalent mutually and also with (50); again, I shall only write the first of them:

$$\varkappa = \frac{i(a_{31} + a_{13}) + (a_{23} - a_{32})}{2\sqrt{a_{11} - a_{22} - a_{33} + 1}},$$
$$\lambda = \frac{i(a_{11} - a_{22} - a_{33} + 1) - (a_{21} + a_{12})}{2\sqrt{a_{11} - a_{22} - a_{33} + 1}}, \tag{51}$$
$$\mu = \frac{i(a_{11} - a_{22} - a_{33} + 1) + (a_{21} + a_{12})}{2\sqrt{a_{11} - a_{22} - a_{33} + 1}},$$
$$\nu = \frac{-i(a_{31} + a_{13}) + (a_{23} - a_{32})}{2\sqrt{a_{11} - a_{22} - a_{33} + 1}}.$$

Similarly as there are $2 \times 4 = 8$ equivalent possibilities for $\varkappa, \lambda, \mu, \nu$, we can derive, by using (46), for the existing 4 results for c_1, c_2, c_3, c_4, further 4 equivalent possibilities for c_1, c_2, c_3, c_4; I refrain from writing them out.

The proof, that the eight possibilities for $\varkappa, \lambda, \mu, \nu$ and c_1, c_2, c_3, c_4 respectively mentioned here are indeed eight equivalent possibilities, can be furnished e. g. on a spatial rotation by expressing a_{ik} with the help of Eulerian angles, i. e. by specializing the equations (38) by putting $\Theta' = \Theta$, $\varphi' = \varphi$, $\omega' = \omega$. So, we obtain the known expressions:

$$\begin{aligned} a_{11} &= \cos\varphi \cos\omega - \sin\varphi \sin\omega \cos\Theta, \\ a_{21} &= -\sin\varphi \cos\omega - \cos\varphi \sin\omega \cos\Theta, \\ a_{31} &= \sin\omega \sin\Theta, \\ a_{12} &= \cos\varphi \sin\omega + \sin\varphi \cos\omega \cos\Theta, \\ a_{22} &= -\sin\varphi \sin\omega + \cos\varphi \cos\omega \cos\Theta, \\ a_{32} &= -\cos\omega \sin\Theta, \end{aligned} \tag{52}$$

$$\begin{aligned} a_{13} &= \sin\varphi \sin\Theta, \\ a_{23} &= \cos\varphi \sin\Theta, \\ a_{33} &= \cos\Theta. \end{aligned}$$

In case that $a_{44} = -1$ we get from (43) the other four parameters $\varkappa', \lambda', \mu', \nu'$ by specializing equations (25) (the first four $\varkappa, \lambda, \mu, \nu$ vanish) as follows:

$$\varkappa' = ic_3 + c_4, \quad \lambda' = ic_1 - c_2,$$
$$\mu' = ic_1 + c_2, \quad \nu' = -ic_3 + c_4;$$

the relations (26) then pass into a single one:

$$\varkappa'\nu' - \lambda'\mu' = 1.$$

This gives a specialization of equations (36)

$$a_{11} = -\tfrac{1}{2}(\varkappa'^2 - \lambda'^2 - \mu'^2 + \nu'^2), \quad a_{12} = -\tfrac{1}{2}i(-\varkappa'^2 + \lambda'^2 - \mu'^2 + \nu'^2),$$
$$a_{13} = -(-\varkappa'\mu' + \lambda'\nu'),$$
$$a_{21} = -\tfrac{1}{2}i(\varkappa'^2 + \lambda'^2 - \mu'^2 - \nu'^2), \quad a_{22} = -\tfrac{1}{2}(\varkappa'^2 + \lambda'^2 + \mu'^2 + \nu'^2),$$
$$a_{23} = i(\varkappa'\mu' + \lambda'\nu'),$$
$$a_{31} = -(-\varkappa'\lambda' + \mu'\nu'), \quad a_{32} = -i(\varkappa'\lambda' + \mu'\nu'), \quad a_{33} = -(\varkappa'\nu' + \lambda'\mu'),$$

which is the same form as (48) except that a_{ik} have here signs opposite to those in (48) and the parameters are dashed. The subsequent procedure is exactly the same as in the case of equations (48); we shall therefore not repeat it here.

Further possibilities of expressing the wanted numbers c_1, c_2, c_3, c_4 are available by specialization of the procedure which starts from the equations (27) to (30); I shall refrain from writing out the corresponding formulae.

7. *General* LORENTZ *transformation with a determinant of* +1. By using (39) and (34) for $c_{ik} = c_i c_k'$ we obtain after an easy arrangement

$$\begin{aligned} 4c_1^2 &= A_{23}^{23} - A_{31}^{31} - A_{12}^{12} + 1 + A_{14}^{23} - A_{24}^{31} - A_{34}^{12} = \\ &= A_{14}^{14} - A_{24}^{24} - A_{34}^{34} + 1 + A_{23}^{14} - A_{31}^{24} - A_{12}^{34}, && \{3\} \\ 4c_4^2 &= A_{23}^{23} + A_{31}^{31} + A_{12}^{12} + 1 + A_{14}^{23} + A_{24}^{31} + A_{34}^{12} = \\ &= A_{14}^{14} + A_{24}^{24} + A_{34}^{34} + 1 + A_{23}^{14} + A_{31}^{24} + A_{12}^{34}, && \{1\} \\ 4c_2c_3 &= A_{12}^{31} + A_{31}^{12} + A_{34}^{31} + A_{24}^{12} = \\ &= A_{34}^{24} + A_{24}^{34} + A_{12}^{24} + A_{31}^{34}, && \{3\} \\ 4c_1c_4 &= A_{12}^{31} - A_{31}^{12} + A_{34}^{31} - A_{24}^{12} = \\ &= A_{34}^{24} - A_{24}^{34} + A_{12}^{24} - A_{31}^{34}. && \{3\} \end{aligned} \tag{53}$$

In this arrangement

$$A_{ns}^{mr} = a_{mn}a_{rs} - a_{ms}a_{rn}; \tag{54}$$

because $\{\det. |a_{ik}|\} = +1$, the determinant A_{ns}^{mr} is equal to its algebraic complement, taken out of the matrix of elements a_{ik}.

We find analogous expressions for $4c_1'^2$, $4c_4'^2$, $4c_2'c_3'$, $4c_1'c_4'$ from (53) by changing the sign of every determinant A^{mr}_{ns}, which comprises only one index 4.

From there we get, if we take into consideration the equations (34), 8 equivalent possibilities for 8 quantities c_k, c_k', $k = 1, 2, 3, 4$.

More generally we carry on similarly as in equations (27)

$$r_j = \beta_{j\rho} c_\rho, \quad (c_1' = c_5,\ c_2' = c_6,\ c_3' = c_7,\ c_4' = c_8), \tag{55}$$

under the assumption that $D = \{\text{det.}\ |\beta_{j\rho}|\} \neq 0$, so that

$$r_j r_k = \tfrac{1}{2}(\beta_{j\rho}\beta_{k\sigma} + \beta_{j\sigma}\beta_{k\rho}) c_\rho c_\sigma; \tag{56}$$

from there we obtain for

$$r_k = \frac{(\beta_{j\varrho}\beta_{k\sigma} + \beta_{j\sigma}\beta_{k\varrho}) c_\varrho c_\sigma}{2\sqrt{\beta_{j\varrho}\beta_{j\sigma} c_\varrho c_\sigma}} \tag{57}$$

8 completely equivalent possibilities which lead to an infinite number of further equivalent possibilities, if we change the coefficients $\beta_{j\rho}$. By further putting

$$r_j' = \beta_{j\rho}' c_\rho \tag{58}$$

and substituting into this relation from the equations (28) and (29)

$$c_\varrho = B_{k\varrho} r_k, \quad B_{k\varrho} = \frac{1}{D}\frac{\partial D}{\partial \beta_{k\varrho}}, \tag{59}$$

we arrive, with regard to (57), at

$$r_j' = \beta_{j\varrho}' B_{k\varrho} \frac{(\beta_{j\varrho}\beta_{k\sigma} + \beta_{j\sigma}\beta_{k\varrho}) c_\varrho c_\sigma}{2\sqrt{\beta_{j\varrho}\beta_{j\sigma} c_\varrho c_\sigma}} = \frac{(\beta_{j\varrho}'\beta_{k\sigma}' + \beta_{j\sigma}'\beta_{k\varrho}') c_\varrho c_\sigma}{2\sqrt{\beta_{k\varrho}'\beta_{k\sigma}' c_\varrho c_\sigma}}; \tag{60}$$

the possibility of expressing r_j', when the coefficients $\beta_{j\rho}$ $\beta_{j\rho}'$, are variable, grows again infinitely, but all these expressions are mutually equivalent.

8. *Some simple equivalent possibilities for determination of coefficients $c_1, \ldots, c_4'$ etc.* Using the formulae of paragraph 7 we obtain in the first place the relations

$$\begin{aligned}
c_1 &= \tfrac{1}{2}K_{\mathrm{I}}(A^{23}_{23} - A^{31}_{31} - A^{12}_{12} + 1 + A^{23}_{14} - A^{31}_{24} - A^{12}_{34}),\\
c_2 &= \tfrac{1}{2}K_{\mathrm{I}}(A^{23}_{31} + A^{31}_{23} + A^{23}_{24} + A^{31}_{14}),\\
c_3 &= \tfrac{1}{2}K_{\mathrm{I}}(A^{12}_{23} + A^{23}_{12} + A^{12}_{14} + A^{23}_{34}),\\
c_4 &= \tfrac{1}{2}K_{\mathrm{I}}(A^{31}_{12} - A^{12}_{31} + A^{31}_{34} - A^{12}_{24}),
\end{aligned}$$

$$\begin{aligned}
c_1' &= \tfrac{1}{2}K_{\mathrm{I}}(a_{11} - a_{22} - a_{33} + a_{44}),\\
c_2' &= \tfrac{1}{2}K_{\mathrm{I}}(a_{12} + a_{21} + a_{34} + a_{43}),\\
c_3' &= \tfrac{1}{2}K_{\mathrm{I}}(a_{31} + a_{13} - a_{24} - a_{42}),\\
c_4' &= \tfrac{1}{2}K_{\mathrm{I}}(a_{23} - a_{32} + a_{14} - a_{41}),
\end{aligned}$$

$$(K_{\mathrm{I}})^{-2} = A^{23}_{23} - A^{31}_{31} - A^{12}_{12} + 1 + A^{23}_{14} - A^{31}_{24} - A^{12}_{34}.$$

By cyclic variation of the indices 1, 2, 3 and I, II, III we obtain 2 further possibilities.

The fourth possibility is:

$$\begin{aligned}
c_1 &= \tfrac{1}{2}K_{\mathrm{IV}}(A^{31}_{12} - A^{12}_{31} + A^{31}_{34} - A^{12}_{24}),\\
c_2 &= \tfrac{1}{2}K_{\mathrm{IV}}(A^{12}_{23} - A^{23}_{12} + A^{12}_{14} - A^{23}_{34}),\\
c_3 &= \tfrac{1}{2}K_{\mathrm{IV}}(A^{23}_{31} - A^{31}_{23} + A^{23}_{24} - A^{31}_{14}),\\
c_4 &= \tfrac{1}{2}K_{\mathrm{IV}}(A^{23}_{23} + A^{31}_{31} + A^{12}_{12} + 1 + A^{23}_{14} + A^{31}_{24} + A^{12}_{34}),
\end{aligned}$$

$$\begin{aligned}
c_1' &= \tfrac{1}{2}K_{\mathrm{IV}}(a_{23} - a_{32} + a_{41} - a_{14}),\\
c_2' &= \tfrac{1}{2}K_{\mathrm{IV}}(a_{31} - a_{13} + a_{42} - a_{24}),\\
c_3' &= \tfrac{1}{2}K_{\mathrm{IV}}(a_{12} - a_{21} + a_{43} - a_{34}),\\
c_4' &= \tfrac{1}{2}K_{\mathrm{IV}}(a_{11} + a_{22} + a_{33} + a_{44}),
\end{aligned}$$

$$(K_{\mathrm{IV}})^{-2} = A^{23}_{23} + A^{31}_{31} + A^{12}_{12} + 1 + A^{23}_{14} + A^{31}_{24} + A^{12}_{34}.$$

From these four possibilities we obtain further four when we substitute on the left-hand side of the equations given in this paragraph the dashed symbols for the undashed ones and undashed symbols for the dashed ones and when, at the same time, we change on the right-hand sides the sign of the determinants A^{mr}_{ns}, which contain the index 4 only once. We have thus gained 8 equivalent possibilities for the coefficients $c_1, \ldots, c_4'$.

In a similar way we can arrive at the formulae for the coefficients C_0, C, C_{23}, C_{31}, C_{12}, C_{14}, C_{24}, C_{34} and for $\varkappa, \lambda, \mu, \nu, \varkappa', \lambda', \mu', \nu'$ as well as for $s_1, s_2, s_3, s_4, s_1', s_2', s_3', s_4'$. For the definitions (20), (23) and (25) apply. If we calculate, with the help of these definitions, 8 squares of the (eight) quantities C_0, C, C_{ik} or $\varkappa, \lambda, \ldots, \varkappa', \lambda', \ldots$ or $s_1, s_2, \ldots, s_1', s_2', \ldots$ and 28 products of two and two of these quantities, we have the possibility of conveniently expressing these wanted quantities by means of the coefficients of a given transformation a_{ik}.

I shall first mention the results for the squares and products of the numbers s:

$$\begin{aligned}
2s_1^2 &= A^{23}_{23} - A^{31}_{31} - A^{12}_{12} + 1 + a_{11} - a_{22} - a_{33} + a_{44},\\
2s_1 s_2 &= A^{23}_{31} + A^{31}_{23} + a_{12} + a_{21},\\
2s_1 s_3 &= A^{12}_{23} + A^{23}_{12} + a_{31} + a_{13},\\
2s_1 s_4 &= A^{31}_{12} - A^{12}_{31} + a_{23} - a_{32}.
\end{aligned}$$

$$\begin{aligned}
2s_1 s_1' &= i(A^{23}_{14} - A^{31}_{24} - A^{12}_{34}),\\
2s_1 s_2' &= i(A^{23}_{24} + A^{31}_{14} - a_{34} - a_{43}),\\
2s_1 s_3' &= i(A^{12}_{14} + A^{23}_{34} + a_{24} + a_{42}),\\
2s_1 s_4' &= i(A^{31}_{34} - A^{12}_{24} + a_{41} - a_{14}),
\end{aligned}$$

$$\begin{aligned}
2s_2^2 &= -A^{23}_{23} + A^{31}_{31} - A^{12}_{12} + 1 - a_{11} + a_{22} - a_{33} + a_{44},\\
2s_2 s_3 &= A^{31}_{12} + A^{12}_{31} + a_{23} + a_{32},\\
2s_2 s_4 &= A^{12}_{23} - A^{23}_{12} + a_{31} - a_{13}.
\end{aligned}$$

$$\begin{aligned}
2s_2 s_1' &= i\ (A^{23}_{24} + A^{31}_{14} + a_{34} + a_{43}),\\
2s_2 s_2' &= i(-A^{23}_{14} + A^{31}_{24} - A^{12}_{34}),\\
2s_2 s_3' &= i\ (A^{31}_{34} + A^{12}_{24} - a_{14} - a_{41}),\\
2s_2 s_4' &= i\ (A^{12}_{14} - A^{23}_{34} + a_{42} - a_{24}).
\end{aligned}$$

$$
\begin{aligned}
2s_3^2 &= -A_{23}^{23} - A_{31}^{31} + A_{12}^{12} + 1 - a_{11} - a_{22} + a_{33} + a_{44},\\
2s_3s_4 &= A_{31}^{23} - A_{23}^{31} + a_{12} - a_{21}.
\end{aligned}
$$

$$
\begin{aligned}
2s_3s_1' &= i\ (A_{14}^{12} + A_{34}^{23} - a_{24} - a_{42}),\\
2s_3s_2' &= i\ (A_{34}^{31} + A_{24}^{12} + a_{14} + a_{41}),\\
2s_3s_3' &= i(-A_{14}^{23} - A_{24}^{31} + A_{34}^{12}),\\
2s_3s_4' &= i\ (A_{24}^{23} - A_{14}^{31} + a_{43} - a_{34}).
\end{aligned}
$$

$$
2s_4^2 = A_{23}^{23} + A_{31}^{31} + A_{12}^{12} + 1 + a_{11} + a_{22} + a_{33} + a_{44}.
$$

$$
\begin{aligned}
2s_4s_1' &= i(A_{34}^{31} - A_{24}^{12} + a_{14} - a_{41}),\\
2s_4s_2' &= i(A_{14}^{12} - A_{34}^{23} + a_{24} - a_{42}),\\
2s_4s_3' &= i(A_{24}^{23} - A_{14}^{31} + a_{34} - a_{43}),\\
2s_4s_4' &= i(A_{14}^{23} + A_{24}^{31} + A_{34}^{12}).
\end{aligned}
$$

$$
\begin{aligned}
2s_1'^2 &= -(A_{23}^{23} - A_{31}^{31} - A_{12}^{12} + 1) + a_{11} - a_{22} - a_{33} + a_{44},\\
2s_1's_2' &= -(A_{31}^{23} + A_{23}^{31}) + a_{12} + a_{21},\\
2s_1's_3' &= -(A_{23}^{12} + A_{12}^{23}) + a_{31} + a_{13},\\
2s_1's_4' &= -(A_{12}^{31} - A_{31}^{12}) + a_{23} - a_{32}.
\end{aligned}
$$

$$
\begin{aligned}
2s_2'^2 &= -(-A_{23}^{23} + A_{31}^{31} - A_{12}^{12} + 1) - a_{11} + a_{22} - a_{33} + a_{44},\\
2s_2's_3' &= -\ (A_{12}^{31} + A_{31}^{12}) + a_{23} + a_{32},\\
2s_2's_4' &= -\ (A_{23}^{12} - A_{12}^{23}) + a_{31} - a_{13}.
\end{aligned}
$$

$$
\begin{aligned}
2s_3'^2 &= -(-A_{23}^{23} - A_{31}^{31} + A_{12}^{12} + 1) - a_{11} - a_{22} + a_{33} + a_{44},\\
2s_3's_4' &= -\ (A_{31}^{23} - A_{23}^{31}) + a_{12} - a_{21}.\\
2s_4'^2 &= -\ (A_{23}^{23} + A_{31}^{31} + A_{12}^{12} + 1) + a_{11} + a_{22} + a_{33} + a_{44}.
\end{aligned}
$$

From there easily follow 8 equivalent possibilities for s_1, s_2, s_3, s_4, s_1', s_2', s_3', s_4' similarly as in the special case in paragraph 6.

In a similar way we obtain the wanted parameters $\varkappa$, λ, μ, ν, $\varkappa'$, λ', μ', ν' from the following table of results:

$$
\begin{aligned}
4\varkappa^2 &= \{(a_{11} + A_{23}^{23}) + (a_{22} + A_{31}^{31})\} + i\,\{(a_{12} + A_{31}^{23}) - (a_{21} + A_{23}^{31})\},\\
4\varkappa\lambda &= -(a_{31} + A_{23}^{12}) - i(a_{32} + A_{31}^{12}),\\
4\varkappa\mu &= -(a_{13} + A_{12}^{23}) + i(a_{23} + A_{12}^{31}),\\
4\varkappa\nu &= (a_{33} + A_{12}^{12}) + (a_{44} + 1).
\end{aligned}
$$

$$
\begin{aligned}
4\varkappa\varkappa' &= (A_{14}^{23} + A_{24}^{31}) - i(A_{14}^{31} - A_{24}^{23}),\\
4\varkappa\lambda' &= (a_{42} - A_{14}^{12}) - i(a_{41} + A_{24}^{12}),\\
4\varkappa\mu' &= (a_{24} - A_{34}^{23}) + i(a_{14} + A_{34}^{31}),\\
4\varkappa\nu' &= A_{34}^{12} - i(a_{34} - a_{43}).
\end{aligned}
$$

$$
\begin{aligned}
4\lambda^2 &= \{-(a_{11} + A_{23}^{23}) + (a_{22} + A_{31}^{31})\} - i\{(a_{12} + A_{31}^{23}) + (a_{21} + A_{23}^{31})\},\\
4\lambda\mu &= (a_{33} + A_{12}^{12}) - (a_{44} + 1),\\
4\lambda\nu &= (a_{13} + A_{12}^{23}) + i(a_{23} + A_{12}^{31}).
\end{aligned}
$$

$$
\begin{aligned}
4\lambda\varkappa' &= -(a_{42} + A_{14}^{12}) + i(a_{41} - A_{24}^{12}),\\
4\lambda\lambda' &= -(A_{14}^{23} - A_{24}^{31}) - i(A_{14}^{31} + A_{24}^{23}),\\
4\lambda\mu' &= A_{34}^{12} - i(a_{34} + a_{43}),\\
4\lambda\nu' &= (a_{24} + A_{34}^{23}) - i(a_{14} - A_{34}^{31}).
\end{aligned}
$$

$$
\begin{aligned}
4\mu^2 &= \{-(a_{11} + A_{23}^{23}) + (a_{22} + A_{31}^{31})\} + i\{(a_{12} + A_{31}^{23}) + (a_{21} + A_{23}^{31})\},\\
4\mu\nu &= (a_{31} + A_{23}^{12}) - i(a_{32} + A_{31}^{12}).
\end{aligned}
$$

$$
\begin{aligned}
4\mu\varkappa' &= -(a_{24} + A_{34}^{23}) - i(a_{14} - A_{34}^{31}),\\
4\mu\lambda' &= A_{34}^{12} + i(a_{34} + a_{43}),\\
4\mu\mu' &= -(A_{14}^{23} - A_{24}^{31}) + i(A_{14}^{31} + A_{24}^{23}),\\
4\mu\nu' &= (a_{42} + A_{14}^{12}) + i(a_{41} - A_{24}^{12}).
\end{aligned}
$$

$$
4\nu^2 = \{(a_{11} + A_{23}^{23}) + (a_{22} + A_{31}^{31})\} - i\{(a_{12} + A_{31}^{23}) - (a_{21} + A_{23}^{31})\}.
$$

$$
\begin{aligned}
4\nu\varkappa' &= A_{34}^{12} + i(a_{34} - a_{43}),\\
4\nu\lambda' &= -(a_{24} - A_{34}^{23}) + i(a_{14} + A_{34}^{31}),\\
4\nu\mu' &= -(a_{42} - A_{14}^{12}) - i(a_{41} + A_{24}^{12}),\\
4\nu\nu' &= (A_{14}^{23} + A_{24}^{31}) + i(A_{14}^{31} - A_{24}^{23}).
\end{aligned}
$$

$$
\begin{aligned}
4\varkappa'^2 &= -\{(a_{11} - A_{23}^{23}) + (a_{22} - A_{31}^{31})\} - i\{(a_{12} - A_{31}^{23}) - (a_{21} - A_{23}^{31})\},\\
4\varkappa'\lambda' &= (a_{31} - A_{23}^{12}) + i(a_{32} - A_{31}^{12}),\\
4\varkappa'\mu' &= (a_{13} - A_{12}^{23}) - i(a_{23} - A_{12}^{31}),\\
4\varkappa'\nu' &= -(a_{33} - A_{12}^{12}) - (a_{44} - 1).
\end{aligned}
$$

$$
\begin{aligned}
4\lambda'^2 &= \{(a_{11} - A_{23}^{23}) - (a_{22} - A_{31}^{31})\} + i\{(a_{12} - A_{31}^{23}) + (a_{21} - A_{23}^{31})\},\\
4\lambda'\mu' &= -(a_{33} - A_{12}^{12}) + (a_{44} - 1),\\
4\lambda'\nu' &= -(a_{13} - A_{12}^{23}) - i(a_{23} - A_{12}^{31}).
\end{aligned}
$$

$$
\begin{aligned}
4\mu'^2 &= \{(a_{11} - A_{23}^{23}) - (a_{22} - A_{31}^{31})\} - i\{(a_{12} - A_{31}^{23}) + (a_{21} - A_{23}^{31})\},\\
4\mu'\nu' &= -(a_{31} - A_{23}^{12}) + i(a_{32} - A_{31}^{12}).
\end{aligned}
$$

$$
4\nu'^2 = -\{(a_{11} - A_{23}^{23}) + (a_{22} - A_{31}^{31})\} + i\{(a_{12} - A_{31}^{23}) - (a_{21} - A_{23}^{31})\}.
$$

The first 8 relations of this table are the result found by Dr M. Brdička with the use of the following representation of matrices γ^1, γ^2, γ^3, γ^4:

$$
\gamma^1 = \begin{pmatrix} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & -i & \cdot \\ \cdot & i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{pmatrix},\
\gamma^2 = \begin{pmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix},\
\gamma^3 = \begin{pmatrix} \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & i \\ i & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot \end{pmatrix},\
\gamma^4 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}.
$$

I refrain from writing out a similar table for C_0^2, C_0C, C_0C_{23}, etc.

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