A Contribution to the Dynamics of the Neutral Helium Atom.

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An attempt has been made, to deal with the helium atom problem in analogy to the hydrogen atom in a „singular" case ($Q_1 =$ Constant) the meaning of which in the case of He is, that the sum of the squares of distances of the electrons from the nucleus is constant (and in the case of H, that the orbit of the electron is circular).

The Hamiltonian function of the Problem of Three Bodies of the helium type is made „hydrogen-like". [Equations (1), (2), (3), *(IV)*.] The derivative of the Hamiltonian function with respect to the time gives zero, because the total energy is constant. Using 7 (of 8) equations of motion, it is easily understood that the assumption $Q_1 =$ Constant can be made. It is shown that for this „singular" case, the energy of the helium atom $W = -2\pi^2 me^4 \widehat{Z}^2/J_2^2$, where $\widehat{Z}$ denotes a function of the quantum numbers defined in article IV.

The general form of the energy of helium atom as a function of the action variables is derived from the Jacobi's theorem of the mean values of the potential and kinetic energy in the case of the Coulombian field. (Eq. (42)). It is pointed out that for the action variables J_k holds a relation analogous to the formula for frequencies $\boldsymbol{\nu}_k = \partial W(J)/\partial J_k$, namely $J_k = \partial \overline{L}(\nu)/\partial \nu_k$.

Rutherford's atom consists of a positively charged nucleus, very small compared to the size of the atom, surrounded by a distribution of electrons, whose number Z equals the nuclear charge. It is often compared to the planetary system with a central „sun". The essential mass of the atom is concentrated in the nucleus.

The hydrogen atom ($Z = 1$) presents no difficulties. With helium ($Z = 2$) various possible ways have been suggested in which the two electrons may be arranged. A most elegant and comprehensive investigation on this subject has been published by M. Born and W. Heisenberg and was anticipated by J. H. van Vleck.[1]) This paper gives theoretical

[1]) M. Born u. W. Heisenberg, Ztschr. f. Phys. *16*, 229, (1923). — M. Born (F. Hund), Vorlesungen über Atommechanik. Berlin (J. Springer), 1925, pp. 327, 334. — J. H. van Vleck, Phil. Mag. (6), *44*, 842—869, 1922.

treatment of the case, where *the distance between the nucleus and the No. 1 electron is permanently smaller than the distance between the nucleus and the No. 2 electron,* and shows that the model gives a *disagreement* with spectroscopic results.

This difficulty strengthened the doubts about this assumption.

Quite recently Heisenberg[2]) has taken a step probably of fundamental importance by formulating the problems of the quantum theory in a novel way by which the difficulties attached to the use of mechanical pictures may, it is hoped, be avoided. Owing to the great difficulties of the mathematical problem involved, it has, however, not yet been possible to apply Heisenberg's theory to questions of atomic structure.[3])

The present paper deals with the same problem as Born and Heisenberg, but quite different theoretical methods have been employed.

We may solve the problem on the classical theory in the following way.

Taking the equations of motion of the Problem of Three Bodies in the case of the neutral helium atom in the form (*I*), it is easily seen that these equations can be reduced by the contact-transformation (*A*) to the form (*II*). Apply to the variables the contact-transformation defined by the equations (*D*). The equations of motion in terms of the new variables are (1), (2), (3) [compare (*IV*)], where H differs from the value of H in the case of the hydrogen atom by the term Z^* [for the hydrogen atom $Z^* = Z = 1$]; from these equations follows the condition (8).

Let us first enquire whether these equations admit of a „*singular*" *solution* (9) *in which the sum of the squares of the distances of the two electrons from the nucleus is permanently constant.*

I.

The equations of the Problem of Three Bodies in the case of the neutral helium atom are

$$\dot{q}_i' = \frac{\partial H}{\partial p_i'}, \qquad \dot{p}_i' = -\frac{\partial H}{\partial q_i'}, \qquad (i = 1,\ 2,\ 3,\ 4),$$

[2]) W. Heisenberg, Ztschr. f. Phys. *33*, 879, (1925). — P. A. M. Dirac, Proceedings of the Royal Society *109*, 642 (1925). — M. Born u. P. Jordan, Ztschr. f. Phys. *34*, 858, (1925). — M. Born, W. Heisenberg u. P. Jordan, Ztschr. f. Phys. *35*, 557 (1926). — N. Bohr, Die Naturwissenschaften, *14*, 1, (1926). Nature, *116*, 845 (1925).

[3]) (Note added in proof. July 1926). — The hydrogen atom has been treated on the new mechanics by P. A. M. Dirac, Proc. Rov Soc., A, *110*, 561—579, 1926, and independentny by W. Pauli jr., ZS. f. Phys. *36*, 336—363, 1926. — Tne new quantum mechanics introduced by Heisenberg was developed from different points of view by various authors. — Cf. Notices respecting new papers of the new quantum mechanics by V. Trkal, Časopis pro pěstování matematiky a fysiky, Praha, *55*, 207, 208, 423, 424, (1926). — See also the work of E. Schrödinger, Ann. d. Phys. *79*, 361—376, 490—257, 734—756, (1926).

where[4])

$$H = \frac{1}{2m}\left(p_1'^2 + \frac{p_3'^2}{q_1'^2}\right) + \frac{1}{2m}\left(p_2'^2 + \frac{p_4'^2}{q_2'^2}\right) - e^2 Z\left(\frac{1}{q_1'} + \frac{1}{q_2'}\right) +$$
$$+ \frac{e^2}{\sqrt{q_1'^2 + q_2'^2 + 2\, q_1'\, q_2'\,(\cos q_3' \cos q_4' + \lambda' \sin q_3' \sin q_4')}} = W, \qquad (I)$$
$$\left(\lambda' = \frac{k^2 - p_3'^2 - p_4'^2}{2\, p_3'\, p_4'}\right),$$

is the Hamiltonian function of the problem. Interpreting these equations, it is easily seen that q_1', q_2' are the distances of the two electrons from the nucleus [thus essentially positive quantities], and $k = \text{const.}$ is the total angular momentum of the system. The Hamiltonian function H represents the total energy $W < 0$ of the system. $Z = 2$ is the so-called „atomic number".

Apply to the variables the contact-transformation defined by the equations

$$x = \frac{\partial V_1}{\partial p_x}, \quad y = \frac{\partial V_1}{\partial p_y}, \quad z = \frac{\partial V_1}{\partial p_z}, \quad u = \frac{\partial V_1}{\partial p_u};$$
$$p_i' = \frac{\partial V_1}{\partial q_i'}, \qquad (i = 1,\ 2,\ 3,\ 4),$$

where[5])

$$V_1 = p_x\, q_1' \cos q_3' + p_y\, q_1' \sin q_3' + p_z\, q_2' \cos q_4' + p_u\, q_2' \sin q_4',$$

i. e.

$$\left.\begin{aligned} &x = q_1' \cos q_3', \quad y = q_1' \sin q_3', && z = q_2' \cos q_4', \quad u = q_2' \sin q_4', \\ &p_1' = p_x \cos q_3' + p_y \sin q_3', && p_3' = q_1'\,(-p_x \sin q_3' + p_y \cos q_3'), \\ &p_2' = p_z \cos q_4' + p_u \sin q_4', && p_4' = q_2'\,(-p_z \sin q_4' + p_u \cos q_4'). \end{aligned}\right\} (A)$$

It follows that

$$q_1' = \sqrt{x^2 + y^2} > 0, \quad p_1'^2 + \frac{p_3'^2}{q_1'^2} = p_x^2 + p_y^2, \quad p_3' = -p_x\, y + p_y\, x,$$
$$q_2' = \sqrt{z^2 + u^2} > 0, \quad p_2'^2 + \frac{p_4'^2}{q_2'^2} = p_z^2 + p_u^2, \quad p_4' = -p_z\, u + p_u\, z.$$

Effecting in H the transformations which have been indicated, we have the new Hamiltonian function

$$H_1 = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2 + p_u^2) - e^2 Z\left(\frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{\sqrt{z^2 + u^2}}\right) +$$
$$+ \frac{e^2}{\sqrt{x^2 + y^2 + z^2 + u^2 + 2\, x\, z + \lambda\, y\, u}} = W, \qquad (11)$$

[4]) M. Born u. W. Heisenberg. Ztschr. f. Phys. *16*, 229, (1923). — E. T. Whittaker, Analytical Dynamics, 2nd ed., Cambridge (University Press) 1917, §§ 155, 157, 158. — In the book of Whittaker, l. c., p. 351, 7 and 8 line from the top, for $-\frac{k^2 - p_3^2 - p_4^2}{2\, p_3\, p_4} \sin q_3 \sin q_4$ read $+\frac{k^2 - p_3^2 - p_4^2}{2\, p_3\, p_4} \sin q_3 \sin q_4$.

[5]) M. Born (F. Hund), Vorlesungen über Atommechanik. Berlin 1925, p. 37. Form. (12).

$$\lambda = \frac{k^2 - (x\,p_y - y\,p_x)^2 - (z\,p_u - u\,p_z)^2}{(x\,p_y - y\,p_x)\,(z\,p_u - u\,p_z)}\,. \qquad (II)$$

The equations of motion now become

$$\dot{x} = \frac{\partial H_1}{\partial p_x}\,, \quad \dot{p}_x = -\frac{\partial H_1}{\partial x}\,, \quad \text{etc.}$$

Let this system be transformed by the contact-transformation

$$p_i = \frac{\partial V_2}{\partial q_i}\,, \quad (i = 1, 2, 3, 4), \quad x = \frac{\partial V_2}{\partial p_x} \quad \text{etc.},$$

where[6])

$$V_2 = (p_y \sin q_4 + p_x \cos q_4)\, q_1 + \sqrt{p_z^2 + (p_y \cos q_4 - p_x \sin q_4)^2}\,.\, q_2 + p_u\,.\, q_3,$$

i. e.

$$x = q_1 \cos q_4 - \frac{p_y \cos q_4 - p_x \sin q_4}{\sqrt{p_z^2 + (p_y \cos q_4 - p_x \sin q_4)^2}}\,.\, q_2 \sin q_4,$$

$$y = q_1 \sin q_4 + \frac{p_y \cos q_4 - p_x \sin q_4}{\sqrt{p_z^2 + (p_y \cos q_4 - p_x \sin q_4)^2}}\,.\, q_2 \cos q_4,$$

$$z = \frac{p_z}{\sqrt{p_z^2 + (p_y \cos q_4 - p_x \sin q_4)^2}}\,.\, q_2,$$

$$u = q_3,$$

$$p_1 = p_y \sin q_4 + p_x \cos q_4,$$

$$p_2 = \sqrt{p_z^2 + (p_y \cos q_4 - p_x \sin q_4)^2},$$

$$p_3 = p_u,$$

$$p_4 = (p_y \cos q_4 - p_x \sin q_4)\left(q_1 - q_2\,\frac{p_y \sin q_4 + p_x \cos q_4}{\sqrt{p_z^2 + (p_y \cos q_4 - p_x \sin q_4)^2}}\right),$$

or

$$\left.\begin{aligned}
p_x &= p_1 \cos q_4 - \frac{p_4}{q_1 p_2 - q_2 p_1}\,.\, p_2 \sin q_4,\\
p_y &= p_1 \sin q_4 + \frac{p_4}{q_1 p_2 - q_2 p_1}\,.\, p_2 \cos q_4,\\
p_z &= p_2 \sqrt{1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}}\,,\\
p_u &= p_3\\
x &= q_1 \cos q_4 - \frac{p_4}{q_1 p_2 - q_2 p_1}\,.\, q_2 \sin q_4,\\
y &= q_1 \sin q_4 + \frac{p_4}{q_1 p_2 - q_2 p_1}\,.\, q_2 \cos q_4,\\
z &= q_2 \sqrt{1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}}\\
u &= q_3.
\end{aligned}\right\} \qquad (B)$$

[6]) As far I know this contact-transformation has not yet been introduced in the litterature; we can get it by the composition of the transformation (C) and (A) of this paper.

We now have

$$p_x^2 + p_y^2 + p_z^2 + p_u^2 = p_1^2 + p_2^2 + p_3^2$$

and

$$x^2 + y^2 + z^2 + u^2 = q_1^2 + q_2^2 + q_3^2,$$

$$x^2 + y^2 = q_1^2 + q_2^2\,\frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}\,, \qquad x\,p_y - y\,p_x = p_4,$$

$$z^2 + u^2 = q_2^2\left(1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}\right) + q_3^2,$$

$$z\,p_u - u\,p_z = (q_2 p_3 - q_3 p_2)\sqrt{1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}}\,.$$

The differential equations, when expressed in terms of the new variables, become

$$\dot{q}_i = \frac{\partial H_2}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H_2}{\partial q_i}\,, \qquad (i = 1, 2, 3, 4),$$

where

$$\left.\begin{aligned}
H_2 = {} & \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - e^2 Z\left\{\left(q_1^2 + q_2^2\,\frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}\right)^{-\frac{1}{2}} + \right.\\
& \left. + \left(q_3^2 + q_2^2\left(1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}\right)\right)^{-\frac{1}{2}}\right\} + \\
& + e^2\left\{q_1^2 + q_2^2 + q_3^2 + 2\,q_2\sqrt{1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}}\left(q_1 \cos q_4 + \right.\right.\\
& \left. - \frac{p_4}{q_1 p_2 - q_2 p_1}\,q_2 \sin q_4\right) + \\
& + q_3\left(q_1 \sin q_4 + \frac{p_4}{q_1 p_2 - q_2 p_1}\,q_2 \cos q_4\right).\\
& \left.\left.\frac{k^2 - p_4^2 - (q_2 p_3 - q_3 p_2)^2\left(1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}\right)}{p_4\,(q_2 p_3 - q_3 p_2)\sqrt{1 - \frac{p_4^2}{(q_1 p_2 - q_2 p_1)^2}}}\right\}^{-\frac{1}{2}}\right.\,.
\end{aligned}\right\} \qquad (III)$$

Perform on the system the contact-transformation defined by the equations

$$q_i = \frac{\partial V_3}{\partial p_i}\,, \; P_i = \frac{\partial V_3}{\partial Q_i}\,, \qquad (i = 1, 2, 3, 4),$$

where[6])

$$V_3 = (p_2 \sin Q_3 + p_1 \cos Q_3)\,.\, Q_1 \cos Q_2 + \\ + \sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}\,.\, Q_1 \sin Q_2 + p_4\, Q_4,$$

i. e.

[6]) E. T. Whittaker, l. c., § 160, p. 349. — J. M. Burgers, Het atoommodel van Rutherford-Bohr (Proefschrift Leiden), Haarlem 1918, § 17, p. 80.

$$
\left.\begin{aligned}
q_1 &= Q_1 \cos Q_2 \cos Q_3 - \frac{p_2 \cos Q_3 - p_1 \sin Q_3}{\sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}}\, Q_1 \sin Q_2 \sin Q_3, \\
q_2 &= Q_1 \cos Q_2 \sin Q_3 + \frac{p_2 \cos Q_3 - p_1 \sin Q_3}{\sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}}\, Q_1 \sin Q_2 \cos Q_3, \\
q_3 &= \frac{p_3}{\sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}}\, Q_1 \sin Q_2, \\
q_4 &= Q_4, \\
P_1 &= (p_2 \sin Q_3 + p_1 \cos Q_3) \cos Q_2 + \\
&\qquad + \sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}\,.\, \sin Q_2, \\
P_2 &= -\,(p_2 \sin Q_3 + p_1 \cos Q_3)\, Q_1 \sin Q_2 + \\
&\qquad + \sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}\,.\, Q_1 \cos Q_2, \\
P_3 &= (p_2 \cos Q_3 - p_1 \sin Q_3)\, Q_1 \cos Q_2 + \\
&\quad - \frac{(p_2 \cos Q_3 - p_1 \sin Q_3)(p_3 \sin Q_3 + p_1 \cos Q_3)}{\sqrt{p_3^2 + (p_2 \cos Q_3 - p_1 \sin Q_3)^2}}\, Q_1 \sin Q_2, \\
P_4 &= p_4,
\end{aligned}\right\}
$$

or finally

$$
\left.\begin{aligned}
p_1 &= \left(P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2\right) \cos Q_3 + \\
&\qquad - \frac{P_3}{P_2}\left(P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2\right) \sin Q_3, \\
p_2 &= \left(P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2\right) \sin Q_3 + \\
&\qquad + \frac{P_3}{P_2}\left(P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2\right) \cos Q_3, \\
p_3 &= \left(P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2\right) \sqrt{1 - \frac{P_3^2}{P_2^2}}\,, \\
p_4 &= P_4, \\
q_1 &= Q_1 \left(\cos Q_2 \cos Q_3 - \frac{P_3}{P_2} \sin Q_2 \sin Q_3\right), \\
q_2 &= Q_1 \left(\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right), \\
q_3 &= Q_1 \sqrt{1 - \frac{P_3^2}{P_2^2}}\, \sin Q_2, \\
q_4 &= Q_4.
\end{aligned}\right\} \quad (C)
$$

We have therefore

$$
p_1^2 + p_2^2 + p_3^2 = P_1^2 + \frac{P_2^2}{Q_1^2}, \quad q_1^2 + q_2^2 + q_3^2 = Q_1^2,
$$

$$
q_1 p_2 - q_2 p_1 = P_3, \quad q_2 p_3 - q_3 p_2 = P_2 \sqrt{1 - \frac{P_3^2}{P_2^2}}\, \sin Q_3.
$$

The equations of motion in terms of the new variables are

$$
\dot{Q}_i = \frac{\partial H_3}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H_3}{\partial Q_i}, \qquad (i = 1, 2, 3, 4),
$$

where, on substitution in H_2 of the new variables for the old, we have

$$
\left.\begin{aligned}
H_3 &= \frac{1}{2\,m}\left(P_1^2 + \frac{P_2^2}{Q_1^2}\right) - \frac{e^2}{Q_1}\, Z^* = W, \\
Z^* &= Z \left[\left\{\left(\cos Q_2 \cos Q_3 - \frac{P_3}{P_2} \sin Q_2 \sin Q_3\right)^2 + \right.\right. \\
&\qquad \left. + \frac{P_4^2}{P_3^2}\left(\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right)^2\right\}^{-\frac{1}{2}} \\
&+ \left\{\left(1 - \frac{P_3^2}{P_2^2}\right) \sin^2 Q_2 + \left(\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right)^2 \left(1 - \frac{P_4^2}{P_3^2}\right)\right\}^{-\frac{1}{2}} \Bigg] \\
&- \Bigg[1 - 2\left\{\frac{P_4}{P_3}\left(\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right)^2 \sin Q_4 + \right. \\
&- \left(\cos Q_2 \cos Q_3 - \frac{P_3}{P_2} \sin Q_2 \sin Q_3\right)\Bigg(\cos Q_2 \sin Q_3 + \\
&\qquad \left. + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\Bigg) \cos Q_4\right\} \sqrt{1 - \frac{P_4^2}{P_3^2}} \\
&+ \frac{k^2 - P_4^2 - P_2^2\left(1 - \frac{P_3^2}{P_2^2}\right)\left(1 - \frac{P_4^2}{P_3^2}\right) \sin^2 Q_3}{P_2 P_4 \sqrt{1 - \frac{P_4^2}{P_3^2}}\, \sin Q_3}\, \sin Q_2 \Bigg\{\Bigg(\cos Q_2 \cos Q_3 + \\
&- \frac{P_3}{P_2} \sin Q_2 \sin Q_3\Bigg) \sin Q_4 + \left(\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right). \\
&\qquad . \frac{P_4}{P_3} \cos Q_4\Bigg\}\Bigg]^{-\frac{1}{2}}.
\end{aligned}\right\} \quad (IV)
$$

The two contact-transformations (B), (C) may be replaced by the contact-transformation of the type

$$
\left.\begin{aligned}
p_x &= \left(P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2\right)\left(\cos Q_3 \cos Q_4 - \frac{P_4}{P_3} \sin Q_3 \sin Q_4\right) + \\
&\quad - \frac{P_3}{P_2}\left(P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2\right)\left(\sin Q_3 \cos Q_4 + \frac{P_4}{P_3} \cos Q_3 \sin Q_4\right), \\
p_y &= \left(P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2\right)\left(\cos Q_3 \sin Q_4 + \frac{P_4}{P_3} \sin Q_3 \cos Q_4\right) + \\
&\quad - \frac{P_3}{P_2}\left(P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2\right)\left(\sin Q_3 \sin Q_4 - \frac{P_4}{P_3} \cos Q_3 \cos Q_4\right), \\
p_z &= \left[P_1\left(\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right) + \right. \\
&\quad \left. - \frac{P_2}{Q_1}\left(\sin Q_2 \sin Q_3 - \frac{P_3}{P_2} \cos Q_2 \cos Q_3\right)\right] \sqrt{1 - \frac{P_4^2}{P_3^2}}, \\
p_u &= \left[P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2\right] \sqrt{1 - \frac{P_3^2}{P_2^2}},
\end{aligned}\right\} \quad (D)
$$

$$
\left.\begin{aligned}
x &= Q_1 \left\{\left(\cos Q_3 \cos Q_4 - \frac{P_4}{P_3} \sin Q_3 \sin Q_4\right) \cos Q_2 + \right. \\
&\qquad \left. - \frac{P_3}{P_2}\left(\sin Q_3 \cos Q_4 + \frac{P_4}{P_3} \cos Q_3 \sin Q_4\right) \sin Q_2 \right\}, \\
y &= Q_1 \left\{\left(\cos Q_3 \sin Q_4 + \frac{P_4}{P_3} \sin Q_3 \cos Q_4\right) \cos Q_2 + \right. \\
&\qquad \left. - \frac{P_3}{P_2}\left(\sin Q_3 \sin Q_4 - \frac{P_4}{P_3} \cos Q_3 \cos Q_4\right) \sin Q_2 \right\}, \\
z &= Q_1 \left\{\cos Q_2 \sin Q_3 + \frac{P_3}{P_2} \sin Q_2 \cos Q_3\right\} \sqrt{1 - \frac{P_4^2}{P_3^2}}, \\
u &= Q_1 \sqrt{1 - \frac{P_3^2}{P_2^2}} \sin Q_2 .
\end{aligned}\right\} (D)
$$

II.

In brief [suppressing the index 3], the equations of motion of the Problem of Three Bodies in the case of the neutral helium atom are reduced to the system of 8th order

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i}, \qquad (i = 1, 2, 3, 4), \tag{1}$$

where

$$H = \frac{1}{2m}\left(P_1^2 + \frac{P_2^2}{Q_1^2}\right) - \frac{e^2 Z^*}{Q_1} = W \tag{2}$$

and

$$Z^* = Z^*\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_2, Q_3, Q_4\right), \quad (P_5 = k = \text{Constant}). \tag{3}$$

Z^* is homogeneous of degree 0 in $P_2, P_3, P_4, P_5 = k$.

Thus, the problem of the helium atom can be reduced to the same form as the problem of the hydrogen atom; the problem of He differs from the corresponding problem of H only by certain modifications in the potential energy [Z^* is in the case of hydrogen $= 1$].

The equations of motion in their new form are

$$
\begin{aligned}
&(4\,a) \quad \dot{Q}_1 = \frac{\partial H}{\partial P_1} = \frac{P_1}{m}, &\quad& \dot{P}_1 = -\frac{\partial H}{\partial Q_1} = \frac{P_2^2}{m Q_1^3} - \frac{e^2 Z^*}{Q_1^2}, \quad (4\,a') \\
&(4\,b) \quad \dot{Q}_2 = \frac{\partial H}{\partial P_2} = \frac{P_2^2}{m Q_1^2} - \frac{e^2}{Q_1}\frac{\partial Z^*}{\partial P_2}, && \dot{P}_2 = -\frac{\partial H}{\partial Q_2} = \frac{e^2}{Q_1}\frac{\partial Z^*}{\partial Q_2}, \quad (4\,b') \\
&(4\,c) \quad \dot{Q}_2 = \frac{\partial H}{\partial P_3} = -\frac{e^2}{Q_1}\frac{\partial Z^*}{\partial P_3}, && \dot{P}_3 = -\frac{\partial H}{\partial Q_3} = \frac{e^2}{Q_1}\frac{\partial Z^*}{\partial Q_3}, \quad (4\,c') \\
&(4\,d) \quad \dot{Q}_4 = \frac{\partial H}{\partial P_4} = -\frac{e^2}{Q_1}\frac{\partial Z^*}{\partial P_4}, && \dot{P}_4 = -\frac{\partial H}{\partial Q_4} = \frac{e^2}{Q_1}\frac{\partial Z^*}{\partial Q_4}. \quad (4\,d')
\end{aligned}
$$

The derivative of Z^* with regard to the time t is

$$\frac{d Z^*}{d t} = \frac{\partial Z^*}{\partial P_2}\dot{P}_2 + \frac{\partial Z^*}{\partial P_3}\dot{P}_3 + \frac{\partial Z^*}{\partial P_4}\dot{P}_4 + \frac{\partial Z^*}{\partial P_5}\dot{P}_5 + \frac{\partial Z^*}{\partial Q_2}\dot{Q}_2 + + \frac{\partial Z^*}{\partial Q_3}\dot{Q}_3 + \frac{\partial Z^*}{\partial Q_4}\dot{Q}_4 .$$

With the help of the equations ($4\,b - d$, $4\,b' - d'$) and by means of the equation $P_5 = k =$ Constant, the simple relation results:

$$\frac{d Z^*}{d t} = \frac{P_2}{m Q_1^2}\frac{\partial Z^*}{\partial Q_2} = \frac{P_2 \dot{P}_2}{m e^2 Q_1}. \tag{5}$$

Differentiating the equation (2) we have

$$\frac{d H}{d t} = \frac{1}{m}\left(P_1 \dot{P}_1 + \frac{P_2 \dot{P}_2}{Q_1^2} - \frac{P_2^2 \dot{Q}_1}{Q_1^3}\right) + + \frac{e^2 Z^*}{Q_1^2}\dot{Q}_1 - \frac{e^2}{Q_1}\frac{d Z^*}{dt} = \frac{d W}{d t} = 0, \tag{6}$$

where W is the constant of energy. With respect to (5) we have from (6) the following equation:

$$\frac{1}{m} P_1 \dot{P}_1 - \left(\frac{P_2^2}{m Q_1^3} - \frac{e^2 Z^*}{Q_1^2}\right) \dot{Q}_1 = 0. \tag{7}$$

By means of the substitution $P_1 = m \dot{Q}_1$ [compare the equation ($4a$)] we obtain

$$\left(\dot{P}_1 - \frac{P_2^2}{m Q_1^3} + \frac{e^2 Z^*}{Q_1^2}\right) \dot{Q}_1 = 0. \tag{8}$$

We arrive at the alternative, that either

$$\dot{P}_1 = \frac{P_2^2}{m Q_1^3} - \frac{e^2 Z^*}{Q_1^2}$$

only, this is the equation ($4\,a'$) of the set ($4\,a - d'$), or also

$$\dot{Q}_1 = 0,$$

so that

$$\underline{Q_1 = \textit{Constant.}} \tag{9}$$

We shall now proceed to consider this „singular" solution (9) *of the considered problem.*

The equations ($4\,a$), (9) shew that

$$P_1 = 0. \tag{10}$$

Thus ($4\,a'$) becomes

$$\dot{P}_1 = \frac{P_2^2}{m Q_1^3} - \frac{e^2 Z^*}{Q_1^2} = 0$$

and we obtain [compare (9)]

$$Q_1 = \frac{P_2^2}{m e^2 Z^*} = \text{Constant}. \tag{11}$$

Substituting from (9), (10), (11) in the Hamiltonian system we may write

$$
\begin{aligned}
&(12\,a)\quad \dot{Q}_1 = 0, && \dot{P}_1 = 0, && (12\,a')\\
&(12\,b)\quad \dot{Q}_2 = \frac{m e^4 Z^*}{P_2^2}\left(\frac{Z^*}{P_2} - \frac{\partial Z^*}{\partial P_2}\right), && \dot{P}_2 = \frac{m e^4 Z^*}{P_2^2}\frac{\partial Z^*}{\partial Q_2}, && (12\,b')\\
&(12\,c)\quad \dot{Q}_3 = -\frac{m e^4 Z^*}{P_2^2}\frac{\partial Z^*}{\partial P_3}, && \dot{P}_3 = \frac{m e^4 Z^*}{P_2^2}\frac{\partial Z^*}{\partial Q_3}, && (12\,c')\\
&(12\,d)\quad \dot{Q}_4 = -\frac{m e^4 Z^*}{P_2^2}\frac{\partial Z^*}{\partial P_4}, && \dot{P}_4 = \frac{m e^4 Z^*}{P_2^2}\frac{\partial Z^*}{\partial Q_4}. && (12\,d')
\end{aligned}
$$

The system ($4\,a$—d') may therefore be replaced in this „singular“ case by the system ($12\,a$—d'). Let H' denote the function derived from H by making the substitution for P_1 and Q_1 [from (10) and (11)], and let s denote any one of the quantities P_2, P_3, P_4, Q_2, Q_3, Q_4; then we have

$$\frac{\partial H'}{\partial s} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial Q_1}\frac{\partial Q_1}{\partial s} = \frac{\partial H}{\partial s} - \dot{P}_1 \frac{\partial Q_1}{\partial s} = \frac{\partial H}{\partial s}$$

and it is therefore allowable to substitute for P_1 and Q_1 in H *before* the derivates of H have been formed. The equations of motion ($12\,a$—d') may be written in the form

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H'}{\partial Q_i}, \quad (i = 2, 3, 4), \tag{12*}$$

where, effecting in H the transformations which have been indicated, we have

$$H' = -\frac{m e^4 Z^{*2}}{2 P_2^2} = W = \text{Constant}. \tag{13}$$

If in (5) we substitute the value of Q_1 from (11) we find

$$\frac{d Z^*}{d t} = \frac{Z^*}{P_2}\dot{P}_2,$$

and therefore

$$\frac{Z^*}{P_2} = \text{Constant}.$$

From (11) and this last equation [or from (11) and (13)] we have

$$Z^* = \text{Constant} \tag{14}$$

and

$$P_2 = \text{Constant}. \tag{15}$$

Refering to the equation ($12\,b'$), we obtain

$$\dot{P}_2 = \frac{m e^4 Z^*}{P_2^2}\frac{\partial Z^*}{\partial Q_2} = 0$$

i. e.

$$\frac{\partial Z^*}{\partial Q_2} = 0. \tag{16}$$

From the latter equation we find

$$Q_2 = Q_2\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_3, Q_4\right) \tag{16*}$$

Now let H'' be the function obtained when this value of Q_2 is substituted in H'; then if s denotes any one of the variables P_2, P_3, P_4, Q_3, Q_4, we have

$$\frac{\partial H''}{\partial s} = \frac{\partial H'}{\partial s} + \frac{\partial H'}{\partial Q_2}\frac{\partial Q_2}{\partial s} = \frac{\partial H'}{\partial s} - \dot{P}_2\frac{\partial Q_2}{\partial s} = \frac{\partial H'}{\partial s};$$

in other words, we can make the substitution for Q_2 in H' *before* forming the derivates of H'; and thus the equations of motion (12*) are reduced to the system

$$\dot{Q}_2 = \frac{\partial H''}{\partial P_2} = \frac{m e^4 Z_1^*}{P_2^2}\left(\frac{Z_1^*}{P_2} - \frac{\partial Z_1^*}{\partial P_2}\right), \tag{17 a}$$

$$\dot{P}_2 = -\frac{\partial H''}{\partial Q_2} = 0, \tag{17 a'}$$

$$\dot{Q}_3 = \frac{\partial H''}{\partial P_3} = -\frac{m e^4 Z_1^*}{P_2^2}\frac{\partial Z_1^*}{\partial P_3}, \tag{17 b}$$

$$\dot{P}_3 = -\frac{\partial H''}{\partial Q_3} = \frac{m e^4 Z_1^*}{P_2^2}\frac{\partial Z_1^*}{\partial Q_3}, \tag{17 b'}$$

$$\dot{Q}_4 = \frac{\partial H''}{\partial P_4} = -\frac{m e^4 Z_1^*}{P_2^2}\frac{\partial Z_1^*}{\partial P_4}, \tag{17 c}$$

$$\dot{P}_4 = -\frac{\partial H''}{\partial Q_4} = \frac{m e^4 Z_1^*}{P_2^2}\frac{\partial Z_1^*}{\partial Q_4}, \tag{17 c'}$$

where

$$H'' = -\frac{m e^4 Z_1^{*2}}{2 P_2^2} = W = \text{Constant} \tag{18}$$

and

$$Z_1^* = Z_1^*\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_3, Q_4\right) \tag{19}$$

is the function obtained when the value of Q_2 from (16*) is substituted in Z^*.

III.

Before proceeding to discuss these equations, it is convenient to calculate the following illustrative example.

The Hamiltonian function H of the problem of two bodies in a plane in the case of the hydrogen atom is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \frac{e^2 Z}{\sqrt{x^2 + y^2}} = W \tag{20}$$

and the equations of motion of this problem are

$$\dot{x} = \frac{\partial H}{\partial p_x}, \qquad \dot{p}_x = -\frac{\partial H}{\partial x},$$

$$\dot{y} = \frac{\partial H}{\partial p_y}, \qquad \dot{p}_y = -\frac{\partial H}{\partial y}.$$

It is easily seen that (x, y) are the coordinates of the electron relative to the nucleus.

Apply to the variables the contact-transformation defined by the equations

$$P_1 = \frac{\partial V_4}{\partial Q_1}, \qquad x = \frac{\partial V_4}{\partial p_x},$$

$$P_2 = \frac{\partial V_4}{\partial Q_2}, \qquad y = \frac{\partial V_4}{\partial p_y},$$

where

$$V_4 = p_x \sqrt{Q_2^2 - Q_1^2} + p_y Q_1.$$

The equations of motion in their new form are

$$\left.\begin{aligned} \dot{Q}_1 &= \frac{\partial H'}{\partial P_1} = \frac{1}{m}\left(P_1 + P_2 \frac{Q_1}{Q_2}\right), \qquad \dot{P}_1 = -\frac{\partial H'}{\partial Q_1} = -\frac{P_1 P_2}{m\, Q_2}, \\ \dot{Q}_2 &= \frac{\partial H'}{\partial P_2} = \frac{1}{m}\left(P_2 + P_1 \frac{Q_1}{Q_2}\right), \\ \dot{P}_2 &= -\frac{\partial H'}{\partial Q_2} = -\frac{1}{Q_2^2}\left(\frac{P_1 P_2 Q_1}{m} - e^2 Z\right), \end{aligned}\right\} \quad (21)$$

where

$$H' = \frac{1}{2\,m}\left(P_1^2 + P_2^2 + 2 P_1 P_2 \frac{Q_1}{Q_2}\right) - \frac{e^2 Z}{Q_2} = W, \quad (Q_2 > 0). \qquad (22)$$

It is well known that $Q_2 =$ Constant [the circular orbit of the electron] is a particular solution of the problem considered.

If $Q_2 =$ Constant then

$$\dot{Q}_2 = \frac{1}{m}\left(P_2 + P_1 \frac{Q_1}{Q_2}\right) = 0,$$

hence

$$P_2 = -P_1 \frac{Q_1}{Q_2}. \qquad (23)$$

Let H'' denote the function derived from H' by making the substitution for P_2 [from (23)], and let s denote any one of the quantities Q_1, Q_2, P_1; then we have

$$\frac{\partial H''}{\partial s} = \frac{\partial H'}{\partial s} + \frac{\partial H'}{\partial P_2}\frac{\partial P_2}{\partial s} = \frac{\partial H'}{\partial s} + \dot{Q}_2 \frac{\partial P_2}{\partial s} = \frac{\partial H'}{\partial s}.$$

The equations of motion may be written in the form

$$\left.\begin{aligned} \dot{Q}_1 &= \frac{\partial H''}{\partial P_1} = \frac{P_1}{m}\left(1 - \frac{Q_1^2}{Q_2^2}\right), \qquad \dot{P}_1 = -\frac{\partial H''}{\partial Q_1} = \frac{P_1^2 Q_1}{m\, Q_2}, \\ \dot{Q}_2 &= \frac{\partial H''}{\partial P_2} = 0, \qquad \dot{P}_2 = -\frac{\partial H''}{\partial Q_2} = -\frac{P_1^2 Q_1^2}{m\, Q_2^3} - \frac{e^2 Z}{Q_2^2}, \end{aligned}\right\} \quad (24)$$

where

$$H'' = \frac{1}{2\,m} P_1^2 \left(1 - \frac{Q_1^2}{Q_2^2}\right) - \frac{e^2 Z}{Q_2} = W. \qquad (25)$$

Differentiating the equation (23) with regard to the time t we have

$$P_1 \dot{Q}_1 + \dot{P}_1 Q_1 + \dot{P}_2 Q_2 = 0.$$

Substituting in this equation for $\dot{Q}_1$, $\dot{P}_1$, $\dot{P}_2$ their values as given by the equations of motion, we have the relation

$$\frac{P_1^2}{m}\left(1 - \frac{Q_1^2}{Q_2^2}\right) - \frac{e^2 Z}{Q_2} = 0. \qquad (26)$$

From the equation (26) we obtain

$$\frac{1}{Q_2} = \frac{1}{2 P_1^2 Q_1^2}\left(-m e^2 Z + \sqrt{m^2 e^4 Z^2 + 4 P_1^4 Q_1^2}\right) \qquad (27)$$

since $Q_2 > 0$. Now let H''' be the function obtained when this value of Q_2 is substituted in H''; then if s denotes any one of the variables P_1, Q_1, we have

$$\frac{\partial H'''}{\partial s} = \frac{\partial H''}{\partial s} + \frac{\partial H''}{\partial Q_2}\frac{\partial Q_2}{\partial s} \neq \frac{\partial H''}{\partial s}.$$

Hence

$$H''' = -\frac{e^2 Z}{4 P_1^2 Q_1^2}\left(-m e^2 Z + \sqrt{m^2 e^4 Z^2 + 4 P_1^4 Q_1^2}\right)$$

is no longer the Hamiltonian function of the considered problem.

The total energy of the system is given by

$$W = -\frac{e^2 Z}{4 P_1^2 Q_1^2}\left(-m e^2 Z + \sqrt{m^2 e^4 Z^2 + 4 P_1^4 Q_1^2}\right)$$

so that

$$P_1 = \sqrt{\frac{2\,m\,(-W)}{1 - \frac{4 W^2}{e^4 Z^2} Q_1^2}}.$$

It is easy to see that

$$\oint \frac{d x}{\sqrt{1 - \left(\frac{x}{\alpha}\right)^2}} = 2\pi |\alpha|.$$

Thus

$$J = \oint P_1\, d Q_1 = 2\pi \frac{e^2 Z}{2|W|} \sqrt{2\,m\,(-W)}$$

and the total energy of the system

$$W = -\frac{2\pi^2 m e^4 Z^2}{J^2}. \tag{28}$$

This is the well-known value of the total energy of the hydrogen atom $[Z = 1,\ J = n h]$.[7])

IV.

We shall now proceed to consider the equations (16*), (17 a—c').

Differentiating the equation (16*) we have

$$\dot{Q}_2 = \frac{\partial Q_2}{\partial P_3}\dot{P}_3 + \frac{\partial Q_2}{\partial P_4}\dot{P}_4 + \frac{\partial Q_2}{\partial Q_3}\dot{Q}_3 + \frac{\partial Q_2}{\partial Q_4}\dot{Q}_4$$

where the quantities $\dot{Q}_2$, $\dot{P}_3$, $\dot{P}_4$, $\dot{Q}_3$, $\dot{Q}_4$ are to be replaced by their values as given by the equations of motion (17 a—c'), i. e.

$$\Phi\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_3, Q_4\right) = 0.$$

From this equation we obtain

$$\text{either } Q_4 = Q_4\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_3\right) \text{ or } Q_3 = Q_3\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_4\right). \tag{29}$$

Let H''' denote the function derived from H'' by making this substitution for Q_4, Q_3 respectively, and let s denote any one of the quantities $P_2, P_3, P_4, Q_3;\ P_2, P_3, P_4, Q_4$ respectively, we have

$$\frac{\partial H'''}{\partial s} = \frac{\partial H''}{\partial s} + \frac{\partial H''}{\partial Q_4}\frac{\partial Q_4}{\partial s} \neq \frac{\partial H''}{\partial s},$$

$$\frac{\partial H'''}{\partial s} = \frac{\partial H''}{\partial s} + \frac{\partial H''}{\partial Q_3}\frac{\partial Q_3}{\partial s} \neq \frac{\partial H''}{\partial s} \quad \text{respectively.}$$

Hence

$$H''' = -\frac{m e^4 Z_2^{*2}}{2 P_2^2} \tag{30}$$

[where

$$Z_2^* = Z_2^*\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_3\right),$$

$$Z_2^* = Z^*_{II}\left(\frac{P_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_4\right), \text{ respectively,} \tag{31}$$

is the function obtained when the value of Q_4, Q_3 respectively is substituted in Z_1^*] is no longer the Hamiltonian function of the considered problem.

[7]) M. Born, l. c. p. 159, form. (3).

The total energy of the system is given by

$$W = -\frac{m e^4 Z_2^{*2}}{2 P_2^2}, \tag{32}$$

where Z_2^* is given by the equations (31).

Let us assume that this equation can be solved with regard to P_3, P_4 respectively; ve get

$$\frac{P_3}{P_2} = f\left(-2 W P_2^2, \frac{P_4}{P_2}, \frac{P_5}{P_2}, Q_3\right),$$

$$\frac{P_4}{P_2} = g\left(-2 W P_2^2, \frac{P_3}{P_2}, \frac{P_5}{P_2}, Q_4\right) \text{ respectively.} \tag{33}$$

If we put

$$P_i = \frac{\partial S}{\partial Q_i}, \quad (i = 2, 3, 4, 5),$$

we see that, if $S = S_2(Q_2) + S_3(Q_3) + S_4(Q_4) + S_5(Q_5)$, then [keeping P_4, P_3 respectively constant]

$$J_3 = \oint P_3\, d Q_3 = \oint \frac{\partial S}{\partial Q_3}\, d Q_3 = \oint \frac{d S_3}{d Q_3}\, d Q_3,$$

$$J_4 = \oint P_4\, d Q_4 = \oint \frac{\partial S}{\partial Q_4}\, d Q_4 = \oint \frac{d S_4}{d Q_4}\, d Q_4 \text{ respectively,} \tag{34}$$

where the quantities P_3, P_4 respectively are to be replaced by their values as given by the equations (33). From the equations (34) we have the relation

$$W = -\frac{m e^4 \hat{Z}^2}{2 P_2^2}, \tag{35}$$

where

$$\hat{Z} = \tilde{Z}\left(\frac{J_3}{P_2}, \frac{P_4}{P_2}, \frac{P_5}{P_2}\right), \hat{Z} = \tilde{Z}'\left(\frac{P_3}{P_2}, \frac{J_4}{P_2}, \frac{J_5}{P_2}\right) \text{ respectively.}$$

Writing

$$J_2 = 2\pi P_2,\ J_3,\ J_4 = 2\pi P_4,\ J_5 = 2\pi P_5,$$

$$J_2 = 2\pi P_2,\ J_3 = 2\pi P_3,\ J_4,\ J_5 = 2\pi P_5 \text{ respectively,}$$

the relation (35) may be written

$$W = -\frac{2\pi^2 m e^4 \hat{Z}^2}{J_2^2}, \tag{36}$$

where

$$\hat{Z} = \hat{Z}\left(\frac{J_3}{J_2}, \frac{J_4}{J_2}, \frac{J_5}{J_2}\right),$$

$$\hat{Z} = \hat{Z}'\left(\frac{J_3}{J_2}, \frac{J_4}{J_2}, \frac{J_5}{J_2}\right) \text{ respectively.} \tag{37}$$

The constants J_k are the action variables of the considered system. The quantum equations of restriction are $J_k = n_k h$, $(k = 2, 3, 4, 5)$.

V.

The equations of motion remain Hamiltonian after this transformation is made and are

$$\dot{w}_k = \frac{\partial W}{\partial J_k}, \quad \dot{J}_k = -\frac{\partial W}{\partial w_k} = 0.$$

Since W is constant, and is a function of the constants J_k only,

$$\dot{w}_k = \nu_k = \frac{\partial W}{\partial J_k} = \text{Const.}^{8)}$$

and

$$w_k = \nu_k t + \delta_k,$$

where ν_k and δ_k are constants. Thus the new coordinates w_k, so introduced, are variables which increase indefinitely with the time, and are therefore called angle variables.

It is well known that the total energy W may be written[9])

$$W = \sum_{k=2}^{5} J_k \dot{w}_k - \overline{L}, \tag{38}$$

where $\overline{L}$ is the mean value of the kinetic potential $L = T - V$, T the kinetic energy and V the potential energy.

According to Jacobi's theorem is in the case of the Coulombian field[10])

$$\left.\begin{aligned} \overline{T} &= -\tfrac{1}{2}\overline{V} \\ W &= \overline{T} + \overline{V} = \tfrac{1}{2}\overline{V} = -\overline{T} \\ \overline{L} &= \overline{T} - \overline{V} = -\tfrac{3}{2}\overline{V} = 3\overline{T} = -3W, \end{aligned}\right\} \tag{39}$$

where $\overline{L}, \overline{W}, \overline{T}, \overline{V}$ etc. are the mean values of L, W, T, V etc.

With the help of these equations, we find the equation[11])

$$\sum_{k=2}^{5} J_k \frac{\partial W}{\partial J_k} = -2W. \tag{40}$$

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8) J. M. Burgers, l. c. § 10, p. 43, equat. (5). — N. Bohr (P. Hertz) Über die Quantentheorie der Linienspektren, Braunschweig, 1923, p. 40, equat. (5).

9) E. T. Whittaker, l. c., § 41, 109. — A. Sommerfeld, Atombau, u. Spektrallinien, 4. Aufl., 1924., Zusatz 4, p. 766, form. (13a). — V. Trkal, Proceedings of the Cambridge Philosophical Society, *21*, 1923 p. 81, form. (3). — J. H. van Vleck, Phys. Review, *22*, 547 (1923).

10) A. Sommerfeld, l. c. Zusatz 5., p. 771, 772. — M. Born, l. c., p. 159 form. (3).

11) Cf. the relation $2\overline{T} = \sum_{k=2}^{5} J_k \nu_k$; compare J. M. Burgers, l. c. 16, p. 72, form. (I). — M. Born, l. c., p. 94, form. (18).

The integral of this partial differential equation is, as it is well known,

$$W = \frac{1}{J_2^2} F\left(\frac{J_3}{J_2}, \frac{J_4}{J_2}, \frac{J_5}{J_2}\right) \tag{41}$$

(where F is the symbol of an arbitrary function), agreeing with formula (36).

Let us write $-\frac{1}{3}\overline{L}$ instead of the energy W, so that we have

$$\sum_{k=2}^{5} J_k \nu_k = \frac{2}{3}\overline{L}, \tag{42}$$

The differential of W is

$$dW = \sum_{k=2}^{5} \frac{\partial W}{\partial J_k} dJ_k = \sum_{k=2}^{5} \nu_k \, dJ_k, \tag{43}$$

i. e.

$$-\frac{1}{3} d\overline{L} = \sum_{k=2}^{5} \nu_k \, dJ_k. \tag{44}$$

From the formula (42) it follows that we can write $d\overline{L}$ in the form

$$\frac{2}{3} d\overline{L} = \sum_{k=2}^{5} J_k \, d\nu_k + \sum_{k=2}^{5} \nu_k \, dJ_k. \tag{45}$$

Thus, on comparing the equations (44) and (45), we have

$$d\overline{L} = \sum_{k=2}^{5} J_k \, d\nu_k. \tag{46}$$

It follows, that[12])

$$J_k = \frac{\partial \overline{L}(\nu)}{\partial \nu_k} = -3\frac{\partial W(\nu)}{\partial \nu_k} \tag{47}$$

where $\overline{L}$, W are functions of the constants ν_k only.

VI.

In brief, it is found that the „singular" solution Q_1 = Constant of the equations of motion of the neutral helium atom (1), (2), (3), (*IV*) leads to the result (36), (37).

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12) (Note added in proof. July 1926). — Recently the author has been able to make an extension of the formula $J_k = \partial \overline{L}(\nu) \,|\, \partial \nu_k$ for each field. In the present paper is this formula deduced for a purely Coulomb field. Cf. V. Trkal, Časopis pro pěstování matematiky a fysiky, *55*, 343—351, 1926.

Since the expression Z^* [cf. the formula (IV)] is very complicated it has not yet been possible to calculate the expression W [cf. (36)] in a more complete form; therefore a confirmation of the theory cannot be given in the present paper. The problem cannot be solved without a knowledge of any convenient approximations.

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