A GENERAL CONDITION FOR THE QUANTISATION OF THE CONDITIONALLY PERIODIC MOTIONS WITH AN APPLICATION FOR THE BOHR ATOM

BY

VICTOR TRKAL, Ph.D.
Lecturer in Theoretical Physics in the Czech University, Prague

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In a conservative dynamical system of k degrees of freedom let $q_1, q_2, \ldots, q_k$ be the generalized Lagrangian coordinates and let L be the kinetic potential: we shall suppose that the constraints are independent of the time, so that L is a given function of the coordinates $q_1, q_2, \ldots, q_k$ and of the velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k$ only, not involving the time t explicitly. If we further introduce momenta, defined in the usual manner as

$$p_r = \frac{\partial L}{\partial \dot{q}_r},$$

the total energy of the system, W, is given as†

$$\sum_{r=1}^{k} p_r \dot{q}_r - L = W = \text{Const.}; \quad L = E_{\text{kin}} - E_{\text{pot}}, \quad W = E_{\text{kin}} + E_{\text{pot}},$$

where E_{kin} and E_{pot} denote the kinetic energy and the potential energy respectively.

Multiply each side of this equation by dt and integrate from the time 0 to T and divide the equation by T. Letting now T increase beyond all limits, we have

$$\lim_{T\to\infty} \left\{ \sum_{r=1}^{k} \frac{1}{T}\int_0^T p_r \dot{q}_r \, dt - \frac{1}{T}\int_0^T L\,dt \right\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T W\,dt = W.$$

In the case of a conditionally periodic motion, it follows that

$$\sum_{r=1}^{k} \frac{1}{T_r}\int_0^{T_r} p_r \dot{q}_r \, dt - \frac{1}{T^*}\int_0^{T^*} L\,dt = W,$$

where T_r and T^* denote the period of the function $p_r\dot{q}_r$ and of the kinetic potential L, respectively. Denoting further the time mean of the kinetic potential

$$\bar{L} = \frac{1}{T^*}\int_0^{T^*} L\,dt,$$

and writing frequencies ν_r instead of periods $\left(\nu_r = \frac{1}{T_r}\right)$, we obtain

$$\sum_{r=1}^{k} \nu_r \int p_r\,dq_r - \bar{L} = W \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

† Cf. E. T. Whittaker, *A treatise on Analytical Dynamics*, 2nd ed., Cambridge, 1917, p. 62.

where $\int p_r\,dq_r = \int_0^{T_r} p_r \dot{q}_r\,dt$ denotes the phase integral. Denote

$$\int p_r\,dq_r = I_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the equation (1) then becomes

$$\sum_{r=1}^{k} I_r \nu_r - \bar{L} = W \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Imagine now I_r, ν_r, $\bar{L}$, W expressed as functions of 'structural' constants (e.g. of matter, charges, field) and 'kinematical' constants. These latter usually specify the shape of the path of the motion considered (e.g. the semi-axis and the numerical excentricity of an elliptical orbit of an electron rotating round a positive nucleus), in other cases they may specify for example the velocity of rotation of a spinning sphere; in the classical theory these 'kinematical' constants can generally acquire any value.

Obviously I_r, ν_r as well as $\bar{L}$ and W are functions of the 'kinematical' quantities $(a, \epsilon, \ldots)$; W, however, can be regarded as a function of $I_1, I_2, \ldots, I_k$, these latter again being functions of $(a, \epsilon, \ldots)$.

Thus we can write

$$\frac{\partial W}{\partial a} = \sum_{r=1}^{k} \frac{\partial W}{\partial I_r}\frac{\partial I_r}{\partial a} = \sum_{r=1}^{k} \nu_r \frac{\partial I_r}{\partial a}, \quad \frac{\partial W}{\partial \epsilon} = \sum_{r=1}^{k} \frac{\partial W}{\partial I_r}\frac{\partial I_r}{\partial \epsilon} = \sum_{r=1}^{k} \nu_r \frac{\partial I_r}{\partial \epsilon}, \text{ etc.} \quad \ldots\ldots (4),$$

since*

$$\frac{\partial W}{\partial I_r} = \nu_r, \quad (r = 1, 2, \ldots, k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Now let us quantise the motion of this dynamical system; then we must use Sommerfeld's condition

$$I_r = \int p_r\,dq_r = n_r h, \quad (r = 1, 2, \ldots, k) \quad \ldots\ldots\ldots (6),$$

where n_r is a positive integer and h Planck's constant. Then $I_1, I_2, \ldots, I_k$ are constants independent of $(a, \epsilon, \ldots)$, so that formula (3) becomes

$$\sum_{r=1}^{k} n_r h \nu_r - \bar{L} = W \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Further, the formula (4) is transformed into

$$\frac{\partial W}{\partial a} = 0, \quad \frac{\partial W}{\partial \epsilon} = 0, \ \ldots \text{ etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

i.e.

$$\delta W = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

I_r being constant and equal to $n_r h$.

* Cf. J. M. Burgers, *Het atoommodel van Rutherford-Bohr.* (Proefschrift.) Haarlem, 1918, p. 43, § 10, equation (5). N. Bohr, "On the Quantum Theory of Line-Spectra. Part I." (*D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturvidensk. og Mathem. Afd.*, 8 Raekke, IV. 1). København, 1918. Separate copy, p. 29, equation (5*).

Substituting into (9) the value W from equation (7), we obtain for the total quantised energy the following condition

$$\delta\left\{\sum_{r=1}^{k} n_r h\nu_r - \bar{L}\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

in which the variation extends to the 'kinematical' constants only. Of course, before variating, we must express the term within the brackets as a function of these 'kinematical' constants.

We notice, that the stationary states of conditionally periodic systems are determined by the condition that the difference between $\sum_{r=1}^{k} n_r h\nu_r$ and the mean kinetic potential $\bar{L}$ should be an extremum (as we shall see from examples, a minimum).

In special relativity-mechanics all the above suppositions remain valid; only the kinetic potential L must be substituted by the modified Lagrangian function

$$L = F - E_{\text{pot}};\ F = -m_0c^2(\sqrt{1-\beta^2} - 1),\ \beta = \frac{v}{c}, \quad \ldots(11),$$

and for the kinetic energy the expression

$$E_{\text{kin}} = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots(12),$$

where v denotes the actual velocity, c the velocity of light and m_0 the mass when at rest.

According to special relativity-mechanics

$$L = F - E_{\text{pot}} = E_{\text{kin}} + F - W, \text{ since } E_{\text{kin}} + E_{\text{pot}} = W \ldots(13).$$

Multiplying each side by dt and integrating from 0 to T, we obtain

$$\int_0^T L\,dt = \int_0^T (E_{\text{kin}} + F)\,dt - WT \quad \ldots\ldots\ldots\ldots(14).$$

Introducing the principal function ('Wirkungsfunktion')

$$S = \int_0^T (E_{\text{kin}} + F)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

we have

$$\frac{1}{T}\left(S - \int_0^T L\,dt\right) = W \quad \ldots\ldots\ldots\ldots\ldots\ldots(16);$$

if the motion is periodic, we see that the following relation must hold

$$\lim_{T\to\infty} \frac{1}{T}\left(S - \int_0^T L\,dt\right) = \bar{S} - \bar{L} = W \quad \ldots\ldots\ldots\ldots(17),$$

where $\bar{S}$ and $\bar{L}$ denote the time means of the functions S and L respectively. Comparison with the relations (3) and (7) gives

$$\bar{S} = \sum_{r=1}^{k} I_r\nu_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

by the classical theory, and

$$\bar{S} = \sum_{r=1}^{k} n_r h\nu_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

by the quantum theory, where

$$\bar{S} = \frac{2}{T^*}\int_0^{T^*} E_{\text{kin}}\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

in ordinary mechanics (since $F = E_{\text{kin}}$) and

$$\bar{S} = \frac{1}{T^*}\int_0^{T^*} (E_{\text{kin}} + F)\,dt \quad \ldots\ldots\ldots\ldots(21),$$

in special relativity mechanics, T^* being the period of the functions behind the integral sign.

Hence we can summarize the chief results as follows:

(1) *The total* (classical) *energy* can be expressed as

$$W = \sum_{r=1}^{k} \nu_r \int p_r\,dq_r - \bar{L} = \sum_{r=1}^{k} I_r\nu_r - \bar{L}.$$

(2) *Its quantisation results from the following variation principle:*

$$\delta\left\{\sum_{r=1}^{k} n_r h\nu_r - \bar{L}\right\} = 0.$$

EXAMPLES.

EXAMPLE 1. An oscillator vibrating linearly about a fixed equilibrium position.

The motion of this (Planck's) oscillator is given by the following known equation

$$m\ddot{\xi} = -k\xi, \quad k > 0,$$

or

$$\ddot{\xi} + 4\pi^2\nu^2\xi = 0,$$

where the constant ν denotes the frequency of this harmonic motion; integrating we obtain

$$\xi = \alpha\cos(2\pi\nu + \vartheta).$$

The kinetic energy is

$$E_{\text{kin}} = \frac{m}{2}\dot{\xi}^2 = 2\pi^2 m\nu^2\alpha^2 \sin^2(2\pi\nu t + \vartheta),$$

and the potential energy

$$E_{\text{pot}} = 2\pi^2 m\nu^2\xi^2 = 2\pi^2 m\nu^2\alpha^2\cos^2(2\pi\nu t + \vartheta).$$

Hence the kinetic potential

$$L = E_{\text{kin}} - E_{\text{pot}} = 2\pi^2 m\nu^2\alpha^2 \cos 2\,(2\pi\nu t + \vartheta)$$

and its time-mean

$$\bar{L} = \nu\int_0^{\frac{1}{\nu}} 2\pi^2 m\nu^2\alpha^2 \cos 2\,(2\pi\nu t + \vartheta)\,dt = 0.$$

Thus we obtain $W = \nu \int p dq = \bar{L} = \nu \int p dq,$

and the quantisation gives $W = nh\nu.$

The only kinematical parameter is ν; the variation of the last expression would have here of course no meaning. Thus the last expression is the definite form for the quantised energy and agrees with the Planck expression*.

Example 2. A rotator spinning round its fixed axis.

The kinetic energy of such a rotator is $E_{\text{kin}} = \frac{1}{2} J \omega^2 = \frac{1}{2} J (2\pi\nu)^2$, where J, ω and ν have their usual significance; this is also the total energy W of the rotator as well as the kinetic potential

$$L = L = W = E_{\text{kin}}.$$

Our general condition takes the form

$$\delta \{nh\nu - L\} = \delta \{nh\nu - \tfrac{1}{2} J (2\pi\nu)^2\} = 0.$$

The only kinematic variable is of course ν. Hence

$$\nu = \frac{nh}{4\pi^2 J}, \quad W = \tfrac{1}{2} J (2\pi\nu)^2 = \frac{n^2 h^2}{8\pi^2 J},$$

as it is well known from other communications*.

Example 3. An electron rotates round a nucleus in a circular orbit.

If we denote by m_0 the mass of the electron, v its velocity, a the radius of its circular orbit, T the period, ν the frequency, $-e$ its charge and E the nuclear charge, we have

$$\nu = \frac{1}{2\pi} \sqrt{\frac{eE}{m_0}}\, a^{-\frac{3}{2}}; \quad W = -\frac{eE}{2a}, \quad E_{\text{kin}} = \frac{eE}{2a}, \quad E_{\text{pot}} = -\frac{eE}{a}.$$

The kinetic potential $L = \bar{L} = \dfrac{3eE}{2a};$

applying our general condition

$$\delta \{nh\nu - \bar{L}\} = \delta \left\{nh \cdot \frac{1}{2\pi} \sqrt{\frac{eE}{m_0}}\, a^{-\frac{3}{2}} - \frac{3eE}{2a}\right\} = 0,$$

we obtain, varying in a, $a = \dfrac{n^2 h^2}{4\pi^2 e E m_0};$

hence the total energy

$$W = -\frac{eE}{2a} = -\frac{2\pi^2 e^2 E^2 m_0}{n^2 h^2},$$

which coincides with the Bohr value†.

* M. Planck, *Vorlesungen über die Theorie der Wärmestrahlung*, 4 Aufl. Leipzig (J. A. Barth), 1921, p. 139, form. (223 *a*), p. 140, form. (231).

† A. Sommerfeld, *Atombau und Spektrallinien*, 2 Aufl. Braunschweig (Fr. Vieweg & Sohn), 1921, p. 243, form. (13).

Example 4. An electron rotates round a nucleus in an elliptical orbit.

The system involves two degrees of freedom and two equal periods (azimuthal and radial)

$$\nu = \nu' = \frac{1}{2\pi} \sqrt{\frac{eE}{m_0}}\, a^{-\frac{3}{2}}.$$

The total energy is $W = -\dfrac{eE}{2a},$

and the time mean of kinetic potential

$$\bar{L} = \frac{3eE}{2a}.$$

In this case the general condition

$$\delta \{nh\nu + n'h\nu' - \bar{L}\} = \delta \{(n + n')\, h\nu - L\}$$

$$= \delta \left\{(n + n')\, h \cdot \frac{1}{2\pi} \sqrt{\frac{eE}{m_0}}\, a^{-\frac{3}{2}} - \frac{3eE}{2a}\right\} = 0$$

gives the semi-axis $a = \dfrac{(n + n')^2 h^2}{4\pi^2 e E m_0},$

and the total energy

$$W = -\frac{eE}{2a} = -\frac{2\pi^2 e^2 E^2 m_0}{(n + n')^2 h^2},$$

coincident with Sommerfeld's calculations*.

Example 5. An electron rotates round a nucleus in a 'relativistic circle.'

In this case the Coulomb's attraction balances the centrifugal force, hence

$$\frac{eE}{a^2} = \frac{mv^2}{a}, \quad \frac{eE}{a} = m_0 c^2 \frac{\dfrac{v^2}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}.$$

Further, the total energy

$$E_{\text{kin}} + E_{\text{pot}} = W = m_0 c^2 \left(\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} - 1\right) - \frac{eE}{a}$$

$$= m_0 c^2 \left\{\sqrt{1 - \frac{v^2}{c^2}} - 1\right\}.$$

But $$\left(\frac{v^2}{c^2}\right)^2 = \left(\frac{eE}{a m_0 c^2}\right)^2 \left(1 - \frac{v^2}{c^2}\right),$$

or $$\frac{v^2}{c^2} = -\frac{1}{2}\left(\frac{eE}{a m_0 c^2}\right)^2 + \sqrt{\frac{1}{4}\left(\frac{eE}{a m_0 c^2}\right)^4 + \left(\frac{eE}{a m_0 c^2}\right)^2},$$

* A. Sommerfeld, *l.c.*, p. 267, form. (20).

and $Z = 1 + \frac{W}{m_0 c^2} = \sqrt{1 - \frac{v^2}{c^2}} = -\frac{eE}{2am_0c^2} + \sqrt{1 + \left(\frac{eE}{2am_0c^2}\right)^2}$.

But $$v = 2\pi a\nu,\ \nu = \frac{c}{2\pi a}\sqrt{1 - Z^2},$$

and $$1 - Z^2 = \frac{eE}{am_0c^2}Z.$$

Hence $$a = \frac{eE}{m_0c^2}\frac{Z}{1 - Z^2},\ \nu = \frac{c}{2\pi}\cdot\frac{c^2}{eE}\frac{(1 - Z^2)^{\frac{3}{2}}}{Z}.$$

The kinetic potential

$$L = F - E_{pot} = -m_0c^2\left(\sqrt{1 - \frac{v^2}{c^2}} - 1\right) + \frac{eE}{a}$$

$$= -m_0c^2(Z - 1) + m_0c^2\frac{1 - Z^2}{Z},$$

and $$\bar{L} = L = m_0c^2\left(1 + \frac{1}{Z} - 2Z\right) \quad \text{..............(22).}$$

Our general condition gives

$$\delta\{nh\nu - \bar{L}\} = \delta\left\{nh\cdot\frac{m_0c^3}{2\pi eE}\cdot\frac{(1 - Z^2)^{\frac{3}{2}}}{Z} - m_0c^2\left(1 + \frac{1}{Z} - 2Z\right)\right\} = 0.$$

Evidently we can vary this expression in Z instead of in a.

Finally we have

$$\sqrt{1 - Z^2} = \frac{2\pi eE}{nhc};\ 1 + \frac{W}{m_0c^2} = Z = \sqrt{1 - \left(\frac{2\pi eE}{nhc}\right)^2},$$

which again is the Bohr expression*.

Example 6. An electron rotates round a nucleus in a 'relativistic ellipse.'

(1) Sommerfeld calculated the total energy in such a case as

$$W = -m_0c^2(Z - 1),\ Z = \sqrt{\frac{p^2 - p_0^2}{p^2 - \epsilon^2 p_0^2}},$$

when the equation of the 'relativistic ellipse' is

$$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon\cos\gamma\phi},$$

and $p_0 = \frac{eE}{c}$, c denoting the velocity of light, $p^2 = \frac{p_0^2}{1 - \gamma^2}$, the semi-axis being

$$a = \frac{\sqrt{p^2 - p_0^2}\sqrt{p^2 - \epsilon^2 p_0^2}}{mcp_0(1 - \epsilon^2)}.$$

Hence we obtain a quadratic equation for p^2

$$p^4 - p_0^2(1 + \epsilon^2)p^2 + \epsilon^2 p_0^4 - a^2m_0^2c^2p_0^2(1 - \epsilon^2)^2 = 0,$$

* A. Sommerfeld, *l.c.*, p. 330, form. (22), where $a = \frac{2\pi e^2}{hc}$.

then

$$p^2 = \tfrac{1}{2}\{p_0^2(1 + \epsilon^2) + p_0\sqrt{p_0^2(1 - \epsilon^2)^2 + 4a^2m_0^2c^2(1 - \epsilon^2)^2}\};$$

further we obtain

$$Z = \sqrt{\frac{p^2 - p_0^2}{p^2 - \epsilon^2 p_0^2}} = -\frac{p_0}{2am_0c} + \sqrt{1 + \left(\frac{p_0}{2am_0c}\right)^2}.$$

According to the binomial theorem we have

$$1 + \frac{W}{m_0c^2} = 1 - \frac{eE}{2am_0c^2} + \frac{1}{2}\frac{e^2E^2}{4a^2m_0^2c^4} + \ldots.$$

Putting $c = \infty$, we obtain $W = -\frac{eE}{2a}$, which is the energy in the non-relativistic case.

(2) The calculation of the two frequencies (radial and azimuthal).

The areal constant

$$p = mr^2\dot{\phi} = \frac{m_0}{\sqrt{1 - \beta^2}}r^2\dot{\phi},\ \beta^2 = \frac{v^2}{c^2},$$

according to Sommerfeld* is

$$\frac{1}{\sqrt{1 - \beta^2}} = Z + \frac{eE}{m_0c^2}\cdot\frac{1}{r},\ Z = 1 + \frac{W}{m_0c^2}.$$

Hence $$p = m_0\left(Z + \frac{p_0}{m_0c}\frac{1}{r}\right)r^2\dot{\phi},\ \frac{1}{\dot{\phi}} = \frac{m_0}{p}\left(Z + \frac{p_0}{m_0c}\frac{1}{r}\right)r^2 \quad (23).$$

But $$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon\cos\psi},\ \psi = \gamma\phi,\ \gamma = \sqrt{1 - \frac{p_0^2}{p^2}},\ p_0 = \frac{eE}{c} \quad (24).$$

If T' denotes the time of revolution counted from a perihelion to the next one, we obtain from the relation

$$dt = \frac{d\phi}{\dot{\phi}} \quad \text{..........................(25),}$$

the period $$T' = \int_0^{\frac{2\pi}{\gamma}}\frac{d\phi}{\dot{\phi}} = \int_0^{2\pi}\frac{d\psi}{\dot{\psi}} = \int_0^{2\pi}\frac{m_0}{\gamma p}\left(Z + \frac{p_0}{m_0c}\frac{1}{r}\right)r^2 d\psi.$$

Substituting the above values we have

$$T' = \left\{a(1 - \epsilon^2)m_0 Z\int_0^{2\pi}\frac{d\psi}{(1 + \epsilon\cos\psi)^2} + \frac{p_0}{c}\int_0^{2\pi}\frac{d\psi}{1 + \epsilon\cos\psi}\right\}\frac{a(1 - \epsilon^2)}{\sqrt{p^2 - p_0^2}}.$$

But $$a = \frac{p_0}{m_0c}\frac{Z}{1 - Z^2},\ \sqrt{p^2 - p_0^2} = \frac{p_0 Z\sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}};$$

$$\int_0^{2\pi}\frac{d\psi}{(1 + \epsilon\cos\psi)^2} = \frac{2\pi}{(1 - \epsilon^2)^{\frac{3}{2}}},\ \int_0^{2\pi}\frac{d\psi}{1 + \epsilon\cos\psi} = \frac{2\pi}{\sqrt{1 - \epsilon^2}}.$$

* *Ann. d. Phys.* 51 (1916), p. 48, form. (B).

Thus
$$T' = \frac{2\pi p_0}{m_0 c^2 (1 - Z^2)^{\frac{3}{2}}}.$$
Replacing the period T' by the frequency ν', we obtain
$$\nu' = \frac{m_0 c^2}{2\pi p_0}(1 - Z^2)^{\frac{3}{2}},$$
where
$$Z = \sqrt{1 + \left(\frac{p_0}{2am_0 c}\right)^2} - \frac{p_0}{2am_0 c}.$$
Hence ν' is a function of the single variable a. But we can also regard ν' as a function of the only variable Z. The second frequency is the reciprocal of the second (azimuthal) period. In increasing the angle $\psi = \gamma\phi$ from 0 to 2π in the time T', the angle $\phi = \frac{\psi}{\gamma}$ increases from 0 to 2π in the time $T = T'\gamma$; putting $T = \frac{1}{\nu}$, $T' = \frac{1}{\nu'}$ we obtain the relation
$$\nu = \frac{\nu'}{\gamma}.$$
But from the expression
$$Z^2 = \frac{p^2 - p_0{}^2}{p^2 - \epsilon^2 p_0{}^2} = \frac{\gamma^2}{1 - \epsilon^2(1 - \gamma^2)},$$
it follows
$$\gamma^2 = \frac{Z^2(1 - \epsilon^2)}{1 - \epsilon^2 Z^2}, \quad 1 - \gamma^2 = \frac{1 - Z^2}{1 - \epsilon^2 Z^2}, \quad p^2 = p_0{}^2 \frac{1 - \epsilon^2 Z^2}{1 - Z^2};$$
hence
$$\nu = \frac{\nu'}{\gamma} = \frac{m_0 c^3}{2\pi e E} \frac{(1 - Z^2)^{\frac{3}{2}}}{Z\sqrt{1 - \epsilon^2}} \sqrt{1 - \epsilon^2 Z^2}.$$
We have thus expressed ν as a function of variables Z and ϵ (or of a and ϵ).

(3) Calculation of the kinetic potential. We have
$$F - m_0 c^2 = m_0 c^2 \sqrt{1 - \beta^2} = -\frac{m_0 c^2 (1 - \epsilon^2) Z}{(1 - Z^2\epsilon^2) + (1 - Z^2)\,\epsilon \cos\psi},$$
$$E_{\text{pot}} = -\frac{eE}{r} = -\frac{m_0 c^2}{1 - \epsilon^2}\frac{1 - Z^2}{Z}(1 + \epsilon\cos\psi),$$
$$L = F - E_{\text{pot}}$$
$$= -\frac{m_0 c^2 (1 - \epsilon^2) Z}{(1 - Z^2\epsilon^2) + (1 - Z^2)\,\epsilon\cos\psi} + \frac{m_0 c^2}{1 - \epsilon^2}\frac{1 - Z^2}{Z}(1 + \epsilon\cos\psi) + m_0 c^2.$$
Putting $\epsilon = 0$, we obtain the kinetic potential of a circle, which agrees with the result in (22).

We have found in (23), (25) the expression for dt. Substituting into (25), (24) the expressions
$$a = \frac{p_0}{m_0 c}\frac{Z}{1 - Z^2}, \quad \sqrt{p^2 - p_0{}^2} = p_0 \frac{Z\sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}},$$

we obtain
$$dt = \frac{p_0}{m_0 c^2}\frac{\sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}}\left\{(1 - \epsilon^2)\frac{Z^2}{1 - Z^2}\frac{d\psi}{(1 + \epsilon\cos\psi)^2} + \frac{d\psi}{1 + \epsilon\cos\psi}\right\},$$
and
$$\int_0^{T'} L\,dt = -p_0 \frac{\sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}}\left[\frac{(1 - \epsilon^2)^2 Z^3}{1 - Z^2}\right.$$
$$\times \int_0^{2\pi} \frac{d\psi}{(1 + \epsilon\cos\psi)^2\{(1 - \epsilon^2 Z^2) + (1 - Z^2)\,\epsilon\cos\psi\}} - Z\int_0^{2\pi}\frac{d\psi}{1 + \epsilon\cos\psi}$$
$$+ (1 - \epsilon^2) Z \int_0^{2\pi} \frac{d\psi}{(1 + \epsilon\cos\psi)\{(1 - \epsilon^2 Z^2) + (1 - Z^2)\,\epsilon\cos\psi\}}$$
$$\left. - \frac{1}{1 - \epsilon^2}\int_0^{2\pi}\frac{1 - Z^2}{Z}\,d\psi\right].$$
Let us calculate some integrals.
$$\text{(I)} \quad \int_0^{2\pi}\frac{d\psi}{1 + \epsilon\cos\psi} = \frac{2\pi}{\sqrt{1 - \epsilon^2}}, \quad |\epsilon| < 1.$$
Putting $\epsilon = \beta : \alpha$, $|\alpha| > |\beta|$, we have
$$\text{(II)} \quad \int_0^{2\pi}\frac{d\psi}{\alpha + \beta\cos\psi} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}}.$$
Differentiating this integral in the parameter α, we obtain
$$\text{(III)} \quad \int_0^{2\pi}\frac{d\psi}{(\alpha + \beta\cos\psi)^2} = 2\pi\alpha(\alpha^2 - \beta^2)^{-\frac{3}{2}}, \quad |\alpha| > |\beta|.$$
Further
$$\text{(IV)} \quad \int_0^{2\pi}\frac{d\psi}{(\alpha + \beta\cos\psi)(\gamma + \delta\cos\psi)}$$
$$= \frac{\beta}{\beta\gamma - \alpha\delta}\int_0^{2\pi}\frac{d\psi}{\alpha + \beta\cos\psi} - \frac{\delta}{\beta\gamma - \alpha\delta}\int_0^{2\pi}\frac{d\psi}{\gamma + \delta\cos\psi}$$
$$= \frac{2\pi}{\beta\gamma - \alpha\delta}\left[\frac{\beta}{\sqrt{\alpha^2 - \beta^2}} - \frac{\delta}{\sqrt{\gamma^2 - \delta^2}}\right]; \quad |\alpha| > |\beta|,\ |\gamma| > |\delta|.$$
Differentiating this integral in the parameter α we obtain
$$\text{(V)} \quad \int_0^{2\pi}\frac{d\psi}{(\alpha + \beta\cos\psi)^2(\gamma + \delta\cos\psi)} = -\frac{2\pi\delta}{(\beta\gamma - \alpha\delta)^2}\left[\frac{\beta}{\sqrt{\alpha^2 - \beta^2}}\right.$$
$$\left. - \frac{\delta}{\sqrt{\gamma^2 - \delta^2}}\right] + \frac{2\pi\alpha\beta}{(\beta\gamma - \alpha\delta)(\alpha^2 - \beta^2)^{\frac{3}{2}}}; \quad |\alpha| > |\beta|,\ |\gamma| > |\delta|.$$
Hence
$$\bar{L} = \frac{1}{T'}\int_0^{T'} L\,dt = +m_0 c^2\left[1 - Z^3 + \frac{1}{\sqrt{1 - \epsilon^2}}\frac{(1 - Z^2)^2}{Z}\right].$$

Our general condition gives

$$\delta\,(nh\nu + n'h\nu' - \bar{L}) = 0.$$

Instead of varying in a and ϵ we may also vary in Z and ϵ. It will be more convenient to use new variables x and q and vary with respect to these. Putting

$$x = \frac{1-Z^2}{Z^2},\; q^2 = \frac{1}{1-\epsilon^2},\; \left(\frac{2\pi e^2}{hc} = \alpha,\; p_0 = \frac{eE}{c}\right),$$

we obtain

$$\delta\left\{\frac{m_0 c^2 e}{\alpha E}\left(\frac{x}{x+1}\right)^{\frac{3}{2}}\left[n\sqrt{1+q^2x}+n'\right] + \frac{m_0 c^2}{(x+1)^{\frac{1}{2}}}(1-q^2x) - m_0 c^2\right\} = 0.$$

The variation in x and in q gives these conditions

$$-\frac{3}{2}\frac{\alpha E}{e}(1-qx^2) + \frac{3}{2}x^{\frac{1}{2}}\left(n\sqrt{1+q^2x}+n'\right)$$

$$+ q\,(x+1)\,xn\left[\frac{x^{\frac{1}{2}}q}{2\sqrt{1+q^2x}} - 2\frac{\alpha E}{ne}\right] = 0 \quad \text{...(26)},$$

$$\frac{q^2x}{1+q^2x} = \left(\frac{\alpha E}{ne}\right)^2 \quad \text{......................(27)}.$$

Substituting (27) into (26), we find

$$x = \frac{1}{Z^2} - 1 = \frac{\left(\frac{\alpha E}{e}\right)^2}{\left[n' + \sqrt{n^2 - \left(\frac{\alpha E}{e}\right)^2}\right]^2},$$

or

$$Z = 1 + \frac{W}{m_0 c^2} = \left\{1 + \frac{\left(\frac{\alpha E}{e}\right)^2}{\left[n' + \sqrt{n^2 - \left(\frac{\alpha E}{e}\right)^2}\right]^2}\right\}^{-\frac{1}{2}},$$

which also is identical with the Sommerfeld* equation.

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* A. Sommerfeld, *l.c.*, p. 330, form. (23); p. 521, form. (5).