A NOTE ON THE HYDRODYNAMICS OF VISCOUS FLUIDS *)

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The cross-product of the velocity and the vorticity in a viscous incompressible fluid is formulated and its properties investigated. When the cross-product is identically null, either the flow is vortex-free, or the velocity and the vorticity are parallel to each other. The second case yields the following important result for three-dimensional flows: if the velocity and the vorticity are related by a position-independent scalar function, that function must be time-independent as well. (English translation of the seventy-five-years old Czech text — Trkal V.: Časopis pro pěstování mathematiky a fysiky *48* (1919) 302–311.)

I.

The Navier-Poisson[1]) hydrodynamics equations in the case of a viscous compressible fluid take on the form [1]

$$
\begin{aligned}
\frac{\mathrm{D}u}{\mathrm{D}t} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\nu\frac{\partial \Theta}{\partial x} + \nu\Delta u,\\
\frac{\mathrm{D}v}{\mathrm{D}t} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{3}\nu\frac{\partial \Theta}{\partial y} + \nu\Delta v,\\
\frac{\mathrm{D}w}{\mathrm{D}t} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{3}\nu\frac{\partial \Theta}{\partial z} + \nu\Delta w,\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \Theta,
\end{aligned}
\tag{1}
$$

where

$$
\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}, \qquad \Delta q = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}.
$$

The quantities occurring in these equations have the following physical meaning: u, v, w are the components of the velocity q of the point (x, y, z) at time t, in the directions of the three rectangular axes of the English co-ordinate system, X, Y, Z are the components of external forces; ρ is the constant density of the fluid, p is the hydrodynamic pressure, and ν is the ratio of the internal friction coefficient of the fluid to its density.

*) The paper has been translated (by I. Gregora) from Czech original which appeared in Časopis pro pěstování mathematiky a fysiky, Vol. 48 (1919), pp. 302–311; see a contextualizing account in the preceding paper in this issue, p. 89. The reprints are available on request from the Editorial Office.

[1]) Nowadays known as Navier-Stokes equations.

If the forces X, Y, Z are conservative, i.e., if they can be derived from a potential Ω, the equations (1) given above can be written in the form

$$\begin{aligned}
\frac{\partial u}{\partial t} - 2v\zeta + 2w\eta &= -\frac{\partial \chi'}{\partial x} + \frac{1}{3}\nu\frac{\partial \Theta}{\partial x} + \nu\Delta u,\\
\frac{\partial v}{\partial t} - 2w\xi + 2u\zeta &= -\frac{\partial \chi'}{\partial y} + \frac{1}{3}\nu\frac{\partial \Theta}{\partial y} + \nu\Delta v,\\
\frac{\partial w}{\partial t} - 2u\eta + 2v\xi &= -\frac{\partial \chi'}{\partial z} + \frac{1}{3}\nu\frac{\partial \Theta}{\partial z} + \nu\Delta w,\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= \Theta,
\end{aligned} \tag{2}$$

where

$$\chi' = \frac{p}{\rho} + \frac{1}{2}q^2 + \Omega, \qquad q^2 = u^2 + v^2 + w^2.$$

Here,

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{3}$$

are the components of the instantaneous angular velocity $\tilde{\omega}$ of an element $d\tau$ of our fluid.

Differentiating the second equation of the system (2) with respect to z, the third equation of the same system with respect to y, and then subtracting the results (thus eliminating χ'), we obtain the first equation of the following system (and quite analogously the other two equations):

$$\begin{aligned}
\frac{\mathrm{D}\xi}{\mathrm{D}t} &= \xi\left(\frac{\partial u}{\partial x} - \Theta\right) + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \nu\Delta\xi,\\
\frac{\mathrm{D}\eta}{\mathrm{D}t} &= \xi\frac{\partial v}{\partial x} + \eta\left(\frac{\partial v}{\partial y} - \Theta\right) + \zeta\frac{\partial v}{\partial z} + \nu\Delta\eta,\\
\frac{\mathrm{D}\zeta}{\mathrm{D}t} &= \xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\left(\frac{\partial w}{\partial z} - \Theta\right) + \nu\Delta\zeta,\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= \Theta, \qquad \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial \zeta}{\partial z} = 0.
\end{aligned} \tag{4}$$

If we put here $\Theta = 0$, we obtain the system of equations for a viscous incompressible fluid. In the case of $v\zeta = w\eta$, $w\xi = u\zeta$, $u\eta = v\zeta$, considering that

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,$$

these equations (4) take on a very simple form:

$$\frac{\partial \xi}{\partial t} = \nu\Delta\xi, \quad \frac{\partial \eta}{\partial t} = \nu\Delta\eta, \quad \frac{\partial \zeta}{\partial t} = \nu\Delta\zeta, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5}$$

For brevity, let us further set

$$\psi = \chi' - \frac{1}{3}\nu\Theta. \tag{6}$$

Then our equations (2) can be written in vectorial form (to simplify notation) as follows:

$$\frac{\partial q}{\partial t} + 2[\tilde{\omega}, q] = \nu\Delta q - \operatorname{grad}\psi, \tag{7}$$

$$\operatorname{div} q = \Theta = \frac{3}{\nu}(\chi' - \psi), \tag{8}$$

where the rotational velocity $\tilde{\omega}$, of components ξ, η, ζ, is connected with the flux velocity q, of components u, v, w, by relations (3), i.e.,

$$\tilde{\omega} = \frac{1}{2}\operatorname{rot} q\,. \tag{9}$$

If we further introduce a new designation for the vectorial product $[\tilde{\omega}, q]$, namely

$$[\tilde{\omega}, q] = \frac{\alpha}{2}, \tag{10}$$

we can write more concisely

$$\frac{\partial q}{\partial t} + \alpha = \nu\Delta q - \operatorname{grad}\psi\,. \tag{11}$$

Using the relations (9) and (10), we convince ourselves readily that the scalar products

$$(\alpha q) = 0, \tag{12}$$

$$(\alpha\tilde{\omega}) = 0, \tag{13}$$

which means that the vector α is perpendicular to the plane defined by the vectors $q, \tilde{\omega}$. If the vector $\tilde{\omega}$ vanishes, or if the directions of the vectors $q, \tilde{\omega}$ coincide, the vector α vanishes, too.

We derive easily some other relations. Thus, e.g., from equation (9) follows the known relation

$$\operatorname{div}\tilde{\omega} = \frac{1}{2}\operatorname{div}\operatorname{rot} q = 0 \tag{14}$$

and further

$$\operatorname{rot}\tilde{\omega} = \frac{1}{2}\operatorname{rot}\operatorname{rot} q = \frac{1}{2}[\operatorname{grad}\operatorname{div} q - \Delta q], \tag{15}$$

whence we obtain

$$\Delta q = \operatorname{grad}\operatorname{div} q - 2\operatorname{rot}\tilde{\omega}. \tag{16}$$

II.

If the vector α vanishes identically and if the potential Ω of the conservative forces X, Y, Z is known, we can find the pressure p; conversely, if the pressure p is known, we can find the potential Ω under the given assumption on the vector α.

The vector α vanishes identically if $\tilde{\omega} = cq$, as shown by eqn. (10). At that, c is a scalar or zero, as the case may be.

If $c = 0$, the vortex velocity $\tilde{\omega}$ vanishes identically; then eqn. (9) yields $\operatorname{rot} q = 0$, whence follows that $q = -\operatorname{grad}\varphi$, i.e., in this case there exists a potential φ of the velocity q. This is in full accordance with the assumption that here we deal with a vortex-free flow.

If c is a scalar and $c \neq 0$, then the vortex lines coincide with the flow lines.[2)]

In the first case, where the velocity has a potential φ, i.e.,

$$q = -\operatorname{grad}\varphi,$$

[2)] The following simple example can convince us that a motion in which the vortex lines coincide with the flow lines is possible, both in a fluid bounded by a convex surface and in a fluid stretching to infinity:

The differential equations of our problem in a viscous incompressible fluid are:

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2cu, \qquad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2cv, \qquad 2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2cw,$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \frac{\partial \xi}{\partial t} = \nu\Delta\xi, \quad \frac{\partial \eta}{\partial t} = \nu\Delta\eta, \quad \frac{\partial \zeta}{\partial t} = \nu\Delta\zeta \qquad \text{(see(5))}.$$

If the fluid fills a sphere centred at $(0,0,0)$ with the radius of $\sqrt{x^2+y^2+z^2}$, and if the velocity component in the direction of the normal n to the surface of the sphere takes on the value

$$u\cos(nx) + v\cos(ny) + w\cos(nz) = q\mathrm{e}^{-4\nu c^2 t}\,\frac{x\sin 2cz + y\cos 2cz}{\sqrt{x^2+y^2+z^2}},$$

where c =const., then the solution to the given problem reads:

$$u = q\mathrm{e}^{-4\nu c^2 t}\sin 2cz, \quad v = q\mathrm{e}^{-4\nu c^2 t}\cos 2cz, \quad w = 0.$$

This will be also the solution in the case where the fluid stretches to infinity, unless we specify boundary conditions. For $\nu = 0$ we obtain the case of a perfect fluid. In every plane parallel to the plane xy the flow velocity then equals q; but in different planes $\parallel$ to xy there is a different direction of the flow (disregarding the fact that it is periodically repeated). At the same time, however, this direction is the axis of the vortex. Since in every point the flow direction coincides with the vortex axis, we deal with a helical motion of the fluid at every point.

As far as I know, this problem was treated for the first time by T. Craig in [2]; an exact proof of the existence of its solution under very general conditions was given by W. Stekloff [3]. A special case of this problem is also solved by G. M. Minchin [4] in his book.

we obtain from the equation (11), which takes on the form

$$\frac{\partial q}{\partial t} = \nu \Delta q - \operatorname{grad} \psi \,, \tag{11'}$$

the relation

$$-\operatorname{grad} \frac{\partial \varphi}{\partial t} = -\nu \operatorname{grad} \Delta q - \operatorname{grad} \psi$$

and further

$$\frac{\partial \varphi}{\partial t} = \nu \Delta \varphi + \psi - T(t),$$

where $T(t)$ is an arbitrary function of time. Hence we find

$$\psi = \chi' + \frac{\nu}{3} \operatorname{div} \operatorname{grad} \varphi = \frac{\partial \varphi}{\partial t} - \nu \Delta \varphi + T(t),$$

and, since

$$\operatorname{div} \operatorname{grad} \varphi = \Delta \varphi$$

we get

$$\chi' = \frac{\partial \varphi}{\partial t} + T(t) - \frac{4}{3} \nu \Delta \varphi$$

and the potential

$$\Omega = \chi' - \frac{p}{\rho} - \frac{1}{2} q^2 = \frac{\partial \varphi}{\partial t} + T(t) - \frac{p}{\rho} - \frac{1}{2} (\operatorname{grad} \varphi)^2 - \frac{4}{3} \nu \Delta \varphi,$$

whence we easily obtain the pressure p.

In the case of a perfect fluid, $\nu = 0$ and the last term vanishes.

Nevertheless, we obtain the same result also for a viscous incompressible fluid. In that case we have

$$\operatorname{div} q = -\operatorname{div} \operatorname{grad} \varphi = -\Delta \varphi = 0,$$

so that a simpler equation holds for pressure p:

$$\Omega = \frac{\partial \varphi}{\partial t} + T(t) - \frac{p}{\rho} - \frac{1}{2} (\operatorname{grad} \varphi)^2 .$$

In the latter case, where the vortex lines coincide with the flow lines, we obtain, as stated above, the relation

$$\tilde{\omega} = cq, \tag{17}$$

where c is a scalar. Hence, with respect to equation (14)

$$\operatorname{div} \tilde{\omega} = \operatorname{div} cq = c \operatorname{div} q + (q, \operatorname{grad} c) = 0 \,; \tag{18}$$

and, comparing equations (9) and (17), we obtain in our special case the relation

$$\operatorname{rot} q = 2cq. \tag{19}$$

Recalling that

$$\operatorname{rot}\tilde{\omega} = \operatorname{rot} cq = c \operatorname{rot} q + [\operatorname{grad} c,\, q] = 2c^2 q - [q,\ \operatorname{grad}\, c], \tag{20}$$

where $(\dots)$ stands for the scalar and $[\dots]$ for vectorial product, then from equation (16) we obtain

$$\Delta q = -4c^2 q + \operatorname{grad}\operatorname{div} q + 2[q,\, \operatorname{grad} c]. \tag{21}$$

Substituting this Δq into equation (11'), we obtain

$$\frac{\partial q}{\partial t} = -4\nu c^2 q - \operatorname{grad}\psi + \nu \operatorname{grad}\operatorname{div} q + 2\nu[q,\, \operatorname{grad} c]. \tag{22}$$

Equation (4) for the case $[\tilde{\omega},\, q] = c$, i.e., $v\zeta = w\eta$, $w\xi = u\zeta$, $u\eta = v\xi$, takes on the form (5), viz.

$$\frac{\partial\tilde{\omega}}{\partial t} = \nu\Delta\tilde{\omega}, \tag{23}$$

and, inserting (17), we find

$$q\frac{\partial c}{\partial t} + c\frac{\partial q}{\partial t} = \nu\Delta(cq) = \nu c\Delta q + \nu q\Delta c + 2\nu(\operatorname{grad} c,\, \operatorname{grad})q.$$

Hence it follows that

$$\frac{\partial q}{\partial t} = \nu\Delta q + \frac{q}{c}\left(\nu\Delta c - \frac{\partial c}{\partial t}\right) + \frac{2\nu}{c}(\operatorname{grad} c,\, \operatorname{grad})q$$

or, with respect to (21)

$$\begin{aligned}\frac{\partial q}{\partial t} &= -\left(4\nu c^2 + \frac{1}{c}\frac{\partial c}{\partial t}\right) q + \nu \operatorname{grad}\operatorname{div} q + 2\nu[q,\, \operatorname{grad} c] \\ &\quad + \frac{\nu q}{c}\Delta c + \frac{2\nu}{c}(\operatorname{grad} c,\, \operatorname{grad})q.\end{aligned} \tag{24}$$

Now, it would be necessary to find from equation (18), e.g. using the equations for flow lines, at least a particular solution for c and substitute this result into (24), integrate this equation and insert the obtained q into eqns. (21) and (22). This will serve us to determine the function c in more detail; in this way we determine more precisely the function q, which must obey equation (23), namely

$$\frac{\partial}{\partial t}\operatorname{rot} q = \nu\Delta \operatorname{rot} q. \tag{23'}$$

Substituting q thus obtained into (22), we find the function ψ; then we determine the function χ', obtaining in turn the potential Ω and hence the pressure p.

In practice, of course, these integrations can be performed only in some very special cases.

Thus, for example, for an incompressible fluid, i.e., if $\operatorname{div} q = 0$, we obtain from equation (18)

$$(q, \operatorname{grad} c) = 0$$

or

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = 0. \tag{18''}$$

However, in view of the meaning of the symbol $\mathrm{D}c/\mathrm{D}t$, equation (18″) can be written in the form

$$\frac{\mathrm{D}c}{\mathrm{D}t} - \frac{\partial c}{\partial t} = 0,$$

whence it is evident that c is a function of a single variable t, provided that none of the components u, v, w equals identically zero. It follows from equations (20), (21), (22), (24) that

$$\operatorname{rot}\tilde{\omega} = 2c^2 q, \tag{20'}$$

$$\Delta q = -4c^2 q, \tag{21'}$$

$$\frac{\partial a}{\partial t} = -4\nu c^2 q - \operatorname{grad}\chi', \tag{22'}$$

$$\frac{\partial q}{\partial t} = -\left(4\nu c^2 + \frac{1}{c}\frac{dc}{dt}\right) q. \tag{24'}$$

From these equations we find that

$$q = \frac{1}{c} f(x, y, z)\, \mathrm{e}^{-\int 4\nu c^2 \mathrm{d}t}$$

and, substituting into (21′)

$$\Delta f = -4c^2 f,$$

whence it is evident that c as a function of a single variable t must reduce to a constant, so that

$$q = g(x, y, z)\, \mathrm{e}^{-4\nu c^2 t}.$$

The function q must obey equation (23'), i.e., g must fulfil the equation

$$\Delta \operatorname{rot} g = -4c^2 \operatorname{rot} g$$

or

$$\operatorname{rot}\,(\Delta g) = \operatorname{rot}\,(-4c^2 g). \tag{23''}$$

If $U(x, y, z)$, $V(x, y, z)$, $W(x, y, z)$ are all bounded functions that have derivatives of the first, second and third order with respect to all the three variables everywhere in the region (D) (the space occupied by the fluid), and satisfy the relations

$$\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} = 2cU, \qquad \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} = 2cV, \qquad \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 2cW,$$

then they satisfy also the relations

$$\Delta U = -4c^2 U, \qquad \Delta V = -4c^2 V, \qquad \Delta W = -4c^2 W.$$

Hence, we can consider them as components of the vector $g(x, y, z)$, because they obey equation (23″). The existence of these functions for the case of a fluid bounded by a closed convex surface (S) on which the velocity component in the direction of the inner normal takes on a given value $F(x, y, z)$ was proved by W. Stekloff ([3], pp. 320, 332). He found that they can be expressed in the following way:

$$\begin{aligned} U &= u_0 + 2cu_1 + (2c)^2 \left(S_1 + \frac{\partial P}{\partial x} \right), \\ V &= v_0 + 2cv_1 + (2c)^2 \left(S_2 + \frac{\partial P}{\partial y} \right), \\ W &= w_0 + 2cw_1 + (2c)^2 \left(S_2 + \frac{\partial P}{\partial z} \right), \end{aligned}$$

where $u_0, v_0, w_0, u_1, v_1, w_1$, are known functions, defined by equations

$$\frac{\partial w_0}{\partial y} - \frac{\partial v_0}{\partial z} = 0, \qquad \frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x} = 0, \qquad \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = 0,$$

$$\frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z} = u_0, \qquad \frac{\partial u_1}{\partial z} - \frac{\partial w_1}{\partial x} = v_0, \qquad \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} = w_0,$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0, \qquad \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0;$$

the normal component

$$\begin{aligned} &u_0 \cos(nx) + v_0 \cos(ny) + w_0 \cos(nz) = F(x, y, z) && \text{on the surface } (S), \\ &u_1 \cos(nx) + v_1 \cos(ny) + w_1 \cos(nz) = 0 && \text{on the surface } (S). \end{aligned}$$

Further,

$$\begin{aligned} S_1 &= \frac{1}{4\pi} \int W_1 \frac{\eta - y}{r^3} \mathrm{d}\tau - \frac{1}{4\pi} \int V_1 \frac{\zeta - z}{r^3} \mathrm{d}\tau, \\ S_2 &= \frac{1}{4\pi} \int U_1 \frac{\zeta - z}{r^3} \mathrm{d}\tau - \frac{1}{4\pi} \int W_1 \frac{\xi - x}{r^3} \mathrm{d}\tau, \\ S_3 &= \frac{1}{4\pi} \int V_1 \frac{\xi - x}{r^3} \mathrm{d}\tau - \frac{1}{4\pi} \int U_1 \frac{\eta - y}{r^3} \mathrm{d}\tau, \end{aligned}$$

where (ξ, η, ζ) runs over all points in the region (D), $\mathrm{d}\tau = \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$, the integrals being taken over the whole region (D) and

$$r^2 = (x - \xi)^2 + (y - \eta)^2 - (z - \zeta)^2.$$

The series U_1, V_1, W_1 converge absolutely and uniformly in the region (D) for all values of the parameter $2c$ smaller than a certain finite number $1/K$, which will be the larger, the smaller are the dimensions of the region (D). It holds then

$$U_1 = \sum_{k=1}^{\infty} (2c)^{k-1} u_k, \qquad V_1 = \sum_{k=1}^{\infty} (2c)^{k-1} v_k, \qquad W_1 = \sum_{k=1}^{\infty} (2c)^{k-1} w_k,$$

$$\frac{\partial w_k}{\partial y} - \frac{\partial v_k}{\partial z} = u_{k-1}, \qquad \frac{\partial u_k}{\partial z} - \frac{\partial w_k}{\partial x} = v_{k-1}, \qquad \frac{\partial v_k}{\partial x} - \frac{\partial u_k}{\partial y} = w_{k-1},$$

$$u_k \cos(nx) + v_k \cos(ny) + w_k \cos(nz) = 0 \qquad \text{on the surface } (S),$$

$$k = 2, 3, \ldots \infty.$$

Finally, P is a harmonic function satisfying the conditions

$$\Delta P = 0 \qquad \text{inside } (D)$$

and the component with respect to the inner normal is defined as

$$\frac{\partial P_1}{\partial n} = \frac{\partial P}{\partial x} \cos(nx) + \frac{\partial P}{\partial y} \cos(ny) + \frac{\partial P}{\partial z} \cos(nz) =$$

$$= -[S_1 \cos(nx) + S_2 \cos(ny) + S_2 \cos(nz)] \qquad \text{on the surface } (S).$$

Hence the components of the velocity q will be

$$u = U e^{-4\nu c^2 t}, \qquad v = V e^{-4\nu c^2 t}, \qquad w = W e^{-4\nu c^2 t} \tag{25}$$

in a fluid bounded by a closed surface (S), on which the velocity component in the direction of the inner normal takes on the given value $e^{-4\nu c^2 t} F(x, y, z)$, if, of course, $|2c| < 1/K$, where K is a constant positive quantity depending only on the properties of the surface (S).

Substituting into (22′) we find that

$$\chi' = \Phi(t)$$

depends only on time; whence the potential

$$\Omega = \chi' - \frac{p}{\rho} - \frac{1}{2} q^2 = \Phi(t) - \frac{p}{\rho} - \frac{1}{2}(U^2 + V^2 + W^2) e^{-8\nu c^2 t},$$

which allows us to calculate the pressure p.

Thus we arrived at the result that in a viscous incompressible fluid, unless at least one of the velocity components is identically equal to zero, c must be a constant and u, v, w are given by formulae (25).

But, also conversely: if $c =$ const., we obtain from equation (23) (taking $\nu = 0$) that $\partial \tilde{\omega} / \partial t = 0$, i.e., the vortex velocity cannot depend on time, hence also c and q are independent of time. If one of the components u, v, w equals zero identically, e.g., $w = 0$, then, in general, c will be a function of the variable z. If two of the components u, v, w are identically equal to zero, the third one is zero, too.

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References

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[4] Minchin G. M.: Uniplanar kinematics of solids and fluids. Oxford, 1882, p. 244.