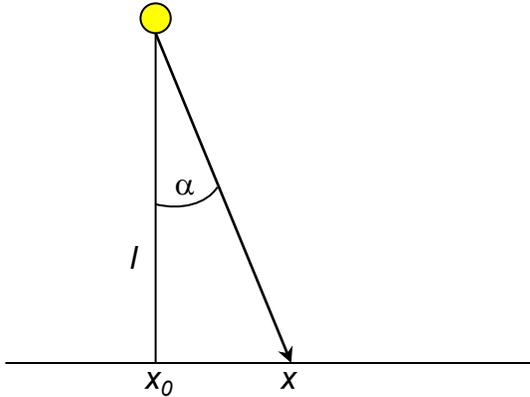


# Problém majáku



$$x - x_0 = l \operatorname{tg}(\alpha)$$

$$\alpha = \operatorname{arctg}\left(\frac{x - x_0}{l}\right) \quad \alpha \in U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$g(x) = \frac{1}{\pi} \left| \frac{d\alpha}{dx} \right| = \frac{1}{\pi} \frac{l}{l^2 + (x - x_0)^2}$$

- předpokládáme, že  $l$  známe
- chceme najít odhad  $x_0$

$$L \equiv P(D_N | x_0) = \prod_{i=1}^N \frac{1}{\pi} \frac{l}{l^2 + (x_i - x_0)^2} \quad (\text{věrohodnost})$$

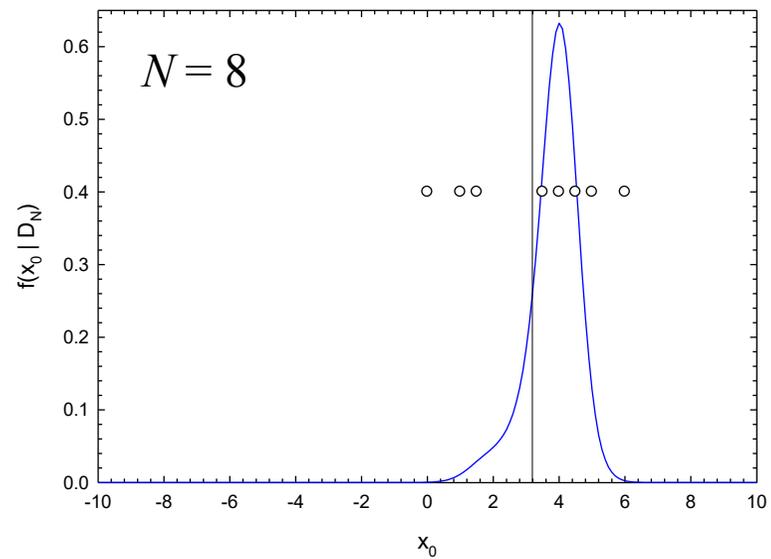
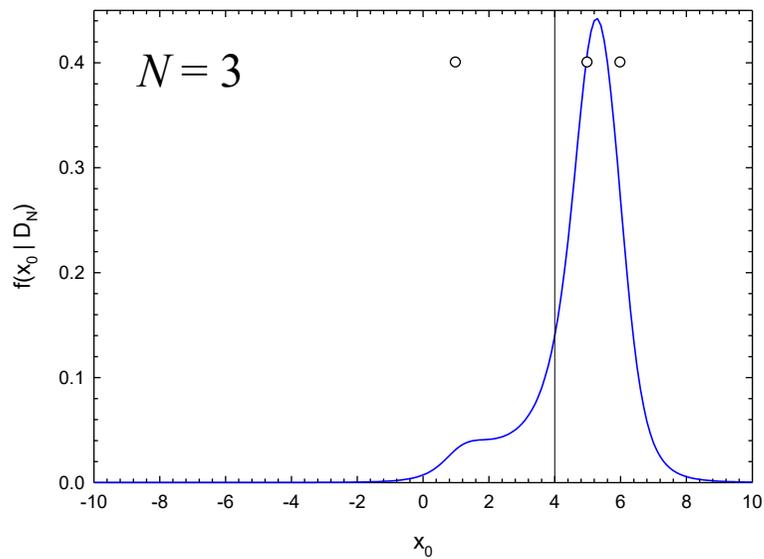
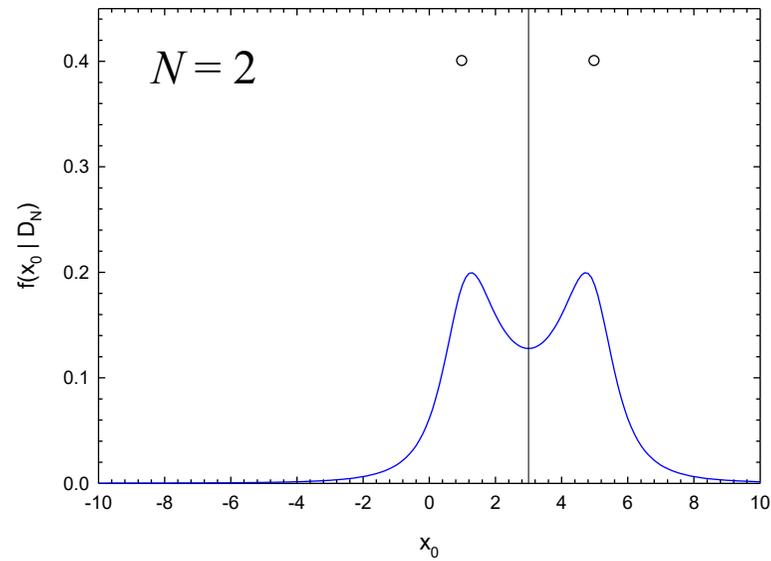
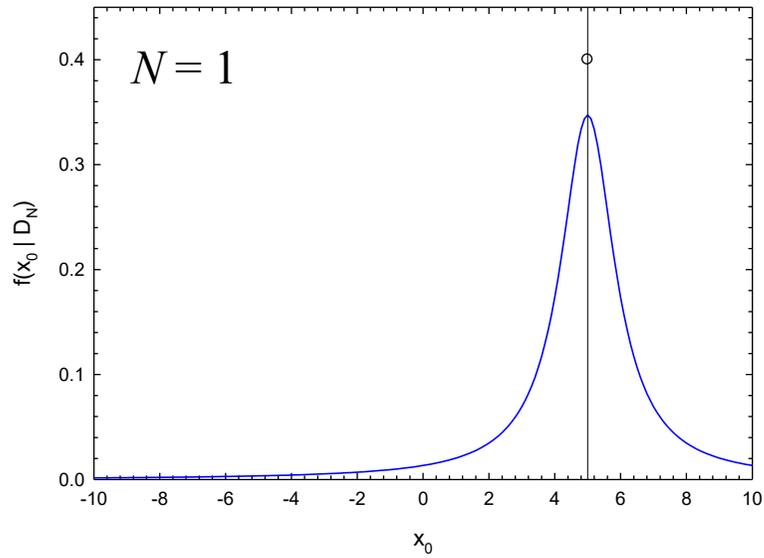
$$f(x_0 | D_N) = \frac{P(D_N | x_0) f(x_0)}{P(D_N)}$$

$$\ln(L) \equiv \sum_{i=1}^N \ln(l) - \sum_{i=1}^N \ln\left(\pi [l^2 + (x_i - x_0)^2]\right)$$

$$f(x_0 | D_N) \propto \prod_{i=1}^N \frac{1}{\pi} \frac{l}{l^2 + (x_i - x_0)^2}$$

$$\text{maximální věrohodnost} \quad \frac{d \ln(L)}{d x_0} = 0 \rightarrow \sum_{i=1}^N \frac{x_i - x_0}{l^2 + (x_i - x_0)^2} = 0$$

# Problém majáku



# Problém majáku

## odhad neurčitosti polohy majáku

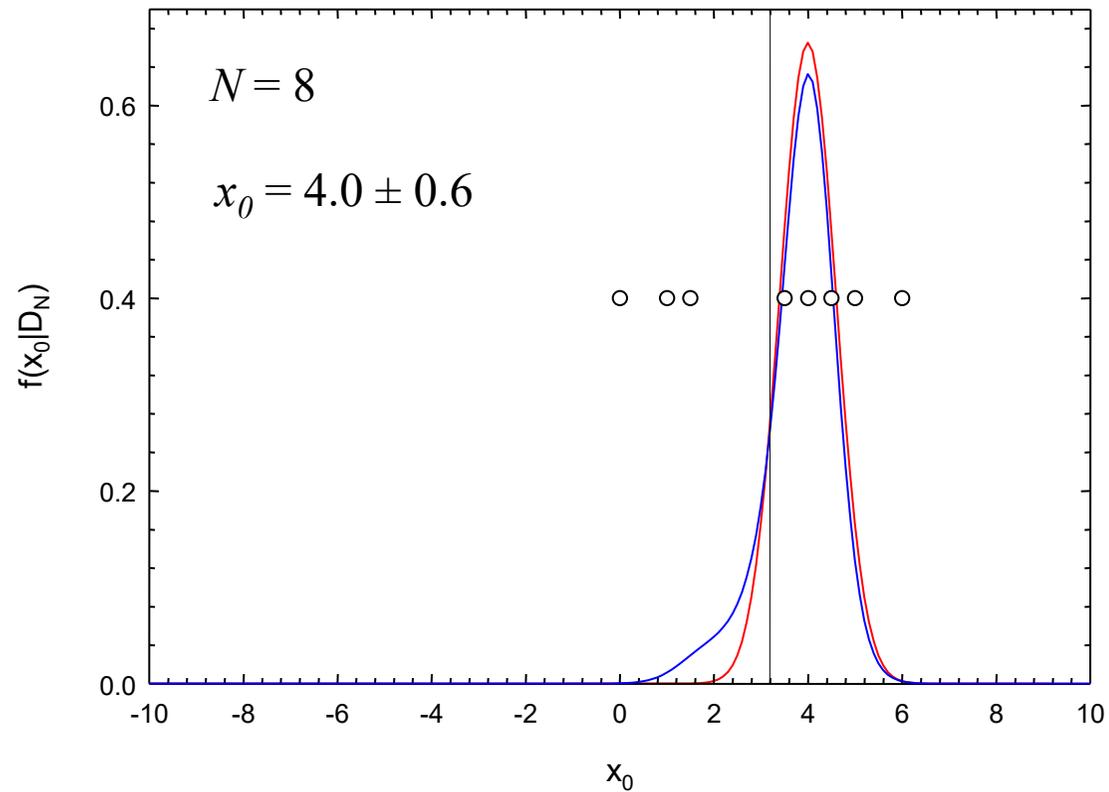
podmínka pro odhad polohy:  $\frac{d \ln(L)}{d x_0} = 0 \rightarrow \sum_{i=1}^N \frac{x_i - x_0}{l^2 + (x_i - x_0)^2} = 0$

podmínka pro odhad neurčitosti:  $\sigma \approx \left( - \frac{d^2 \ln(L)}{d x_0^2} \Big|_{x_0=x_m} \right)^{-\frac{1}{2}}$

$$\sigma \approx \left( 2 \sum_{i=1}^N \frac{l^2 - (x_i - x_m)^2}{[l^2 + (x_i - x_m)^2]^2} \right)^{-\frac{1}{2}}$$

# Problém majáku

$$\sigma \approx \left( 2 \sum_{i=1}^N \frac{l^2 - (x_i - x_m)^2}{[l^2 + (x_i - x_m)^2]^2} \right)^{-\frac{1}{2}}$$

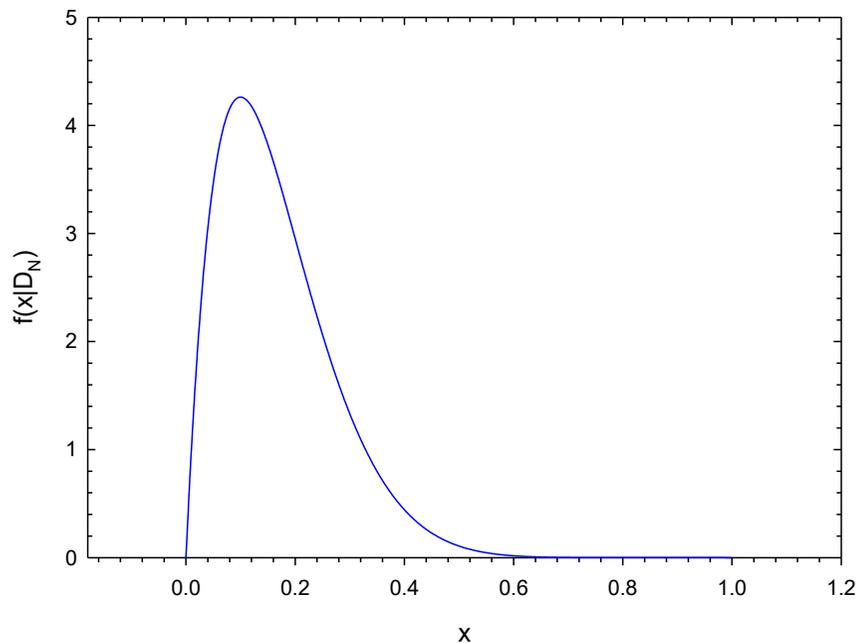


# Interval spolehlivosti

## asymetrická posteriorní pdf

odhad neurčitosti: 
$$\sigma_+ \approx \left( - \frac{d_+^2 \ln L}{d x_0^2} \Big|_{x_0=x_m} \right)^{-\frac{1}{2}} \quad \sigma_- \approx \left( - \frac{d_-^2 \ln L}{d x_0^2} \Big|_{x_0=x_m} \right)^{-\frac{1}{2}}$$

95% interval spolehlivosti  $(x_1, x_2)$ : 
$$P(x \in \langle x_1, x_2 \rangle) = \int_{x_1}^{x_2} f(x|D_N) dx = 0.95 \quad \langle x_1, x_2 \rangle \text{ min}$$

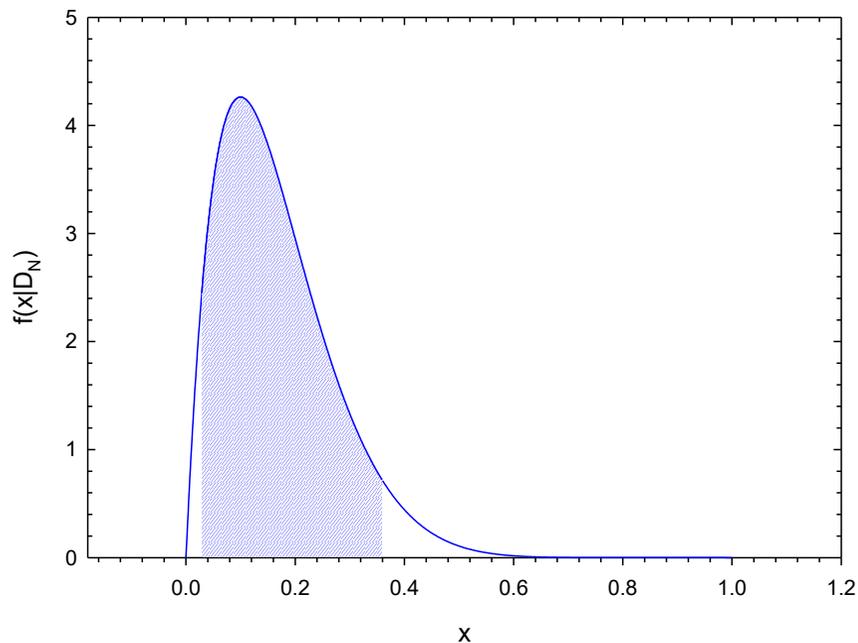


# Interval spolehlivosti

## asymetrická posteriorní pdf

odhad neurčitosti: 
$$\sigma_+ \approx \left( - \frac{d_+^2 \ln L}{d x_0^2} \Big|_{x_0=x_m} \right)^{-\frac{1}{2}} \quad \sigma_- \approx \left( - \frac{d_-^2 \ln L}{d x_0^2} \Big|_{x_0=x_m} \right)^{-\frac{1}{2}}$$

95% interval spolehlivosti  $(x_1, x_2)$ : 
$$P(x \in \langle x_1, x_2 \rangle) = \int_{x_1}^{x_2} f(x|D_N) dx = 0.95 \quad \langle x_1, x_2 \rangle \text{ min}$$



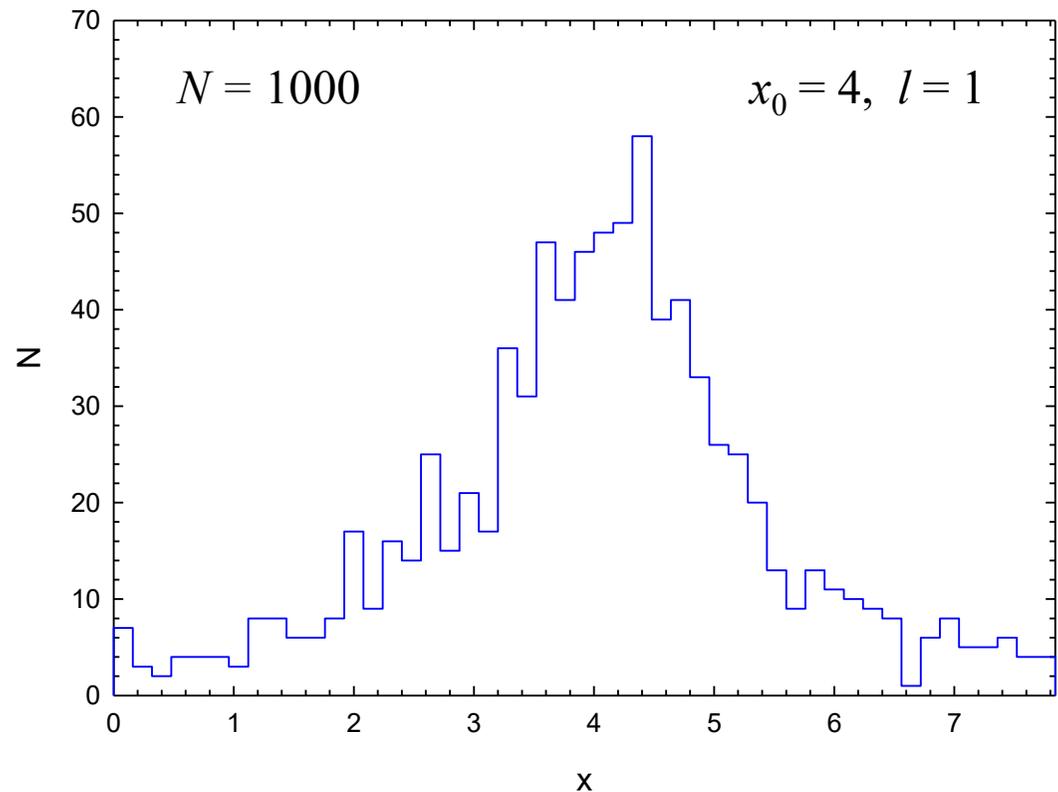
# Problém majáku

**odhad polohy majáku ( $x_0, l$ )**

$$L \equiv P(D_N | x_0) = \prod_{i=1}^N \frac{1}{\pi} \frac{l}{l^2 + (x_i - x_0)^2} \quad (\text{věrohodnost})$$

$$\ln(L) \equiv \sum_{i=1}^N \ln(l) - \sum_{i=1}^N \ln(\pi [l^2 + (x - x_0)^2])$$

- maximum L  $\frac{\partial \ln(L)}{\partial x_0} = 0$   
 $\frac{\partial \ln(L)}{\partial l} = 0$



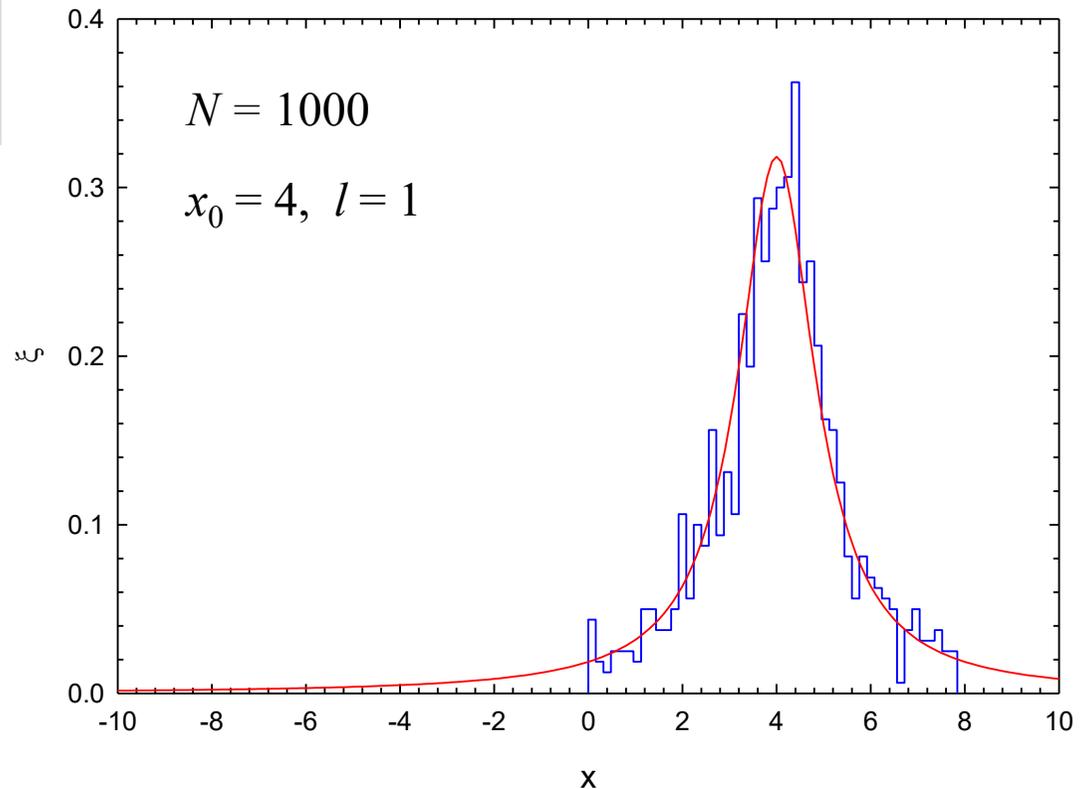
# Problém majáku

## odhad polohy majáku $(x_0, l)$

$$L \equiv P(D_N | x_0) = \prod_{i=1}^N \frac{1}{\pi} \frac{l}{l^2 + (x_i - x_0)^2} \quad (\text{věrohodnost})$$

$$\ln(L) \equiv \sum_{i=1}^N \ln(l) - \sum_{i=1}^N \ln\left(\pi \left[l^2 + (x - x_0)^2\right]\right)$$

- maximum L  $\frac{\partial \ln(L)}{\partial x_0} = 0$   
 $\frac{\partial \ln(L)}{\partial l} = 0$



# Problém majáku

odhad polohy majáku  $(x_0, l)$

$$N = 1000$$

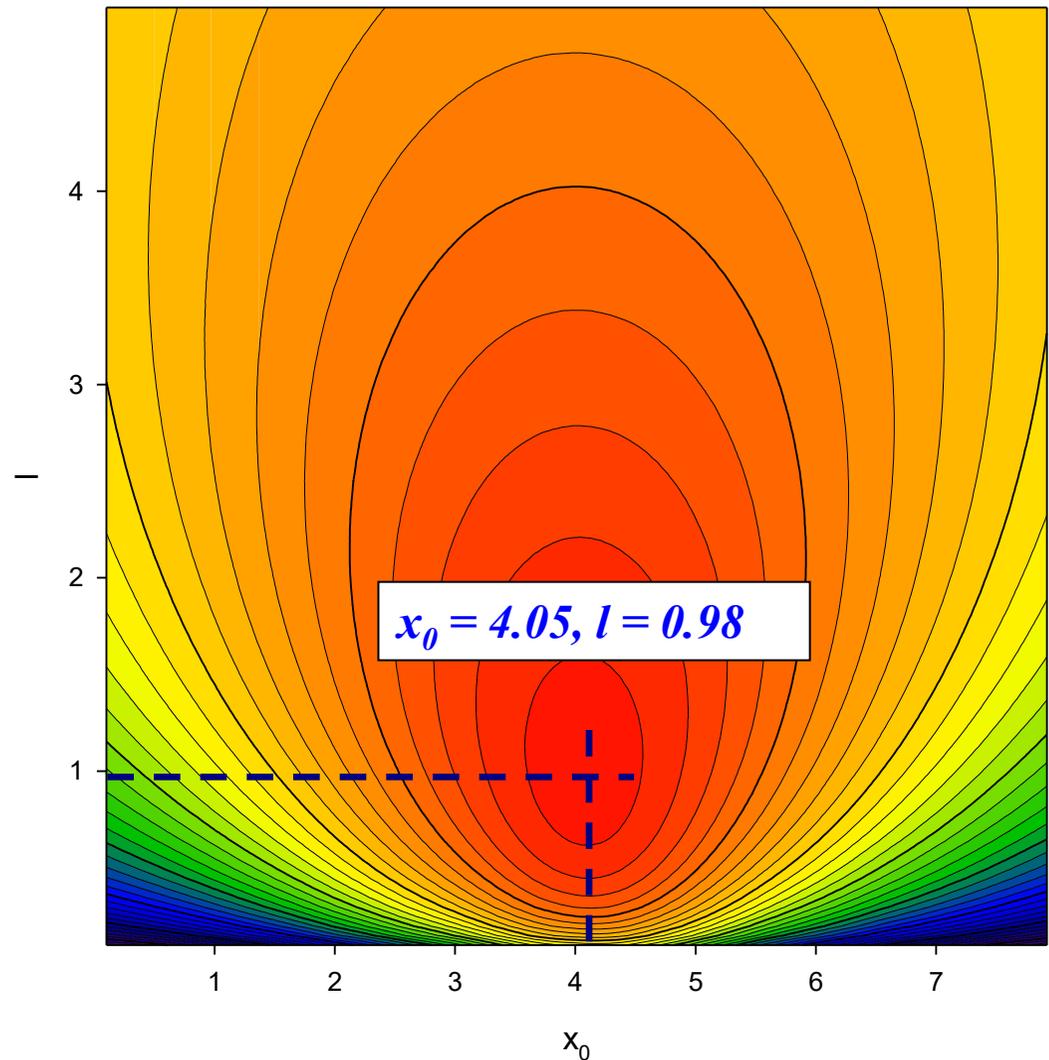
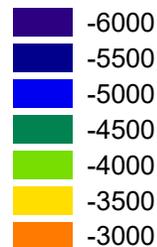
$$x_0 = 4, l = 1$$

$$L \equiv P(D_N | x_0) = \prod_{i=1}^N \frac{1}{\pi} \frac{l}{l^2 + (x_i - x_0)^2}$$

$$\ln(L) \equiv \sum_{i=1}^N \ln(l) - \sum_{i=1}^N \ln(\pi [l^2 + (x_i - x_0)^2])$$

• maximum L

$$\frac{\partial \ln(L)}{\partial x_0} = 0$$
$$\frac{\partial \ln(L)}{\partial l} = 0$$



# Problém majáku – odhad neurčitosti

## odhad polohy majáku $(x_0, l)$

$$L \equiv P(D_N | x_0) = \prod_{i=1}^N \frac{1}{\pi} \frac{l}{l^2 + (x_i - x_0)^2} \quad \longrightarrow \quad \ln(L) = \sum_{i=1}^N \ln l - \sum_{i=1}^N \ln [\pi (l^2 + (x_i - x_0)^2)]$$

- Taylorův rozvoj  $\ln(L)$  v bodě  $[x_{0,m}, l_m]$ :

$$\ln(L) \approx \ln(L(x_{0,m}, l_m)) + \frac{1}{2} \left[ \frac{\partial^2 \ln(L)}{\partial x_0^2} \Big|_{x_{0,m}, l_m} (x_0 - x_{0,m})^2 + \frac{\partial^2 \ln(L)}{\partial l^2} \Big|_{x_{0,m}, l_m} (l - l_m)^2 + 2 \frac{\partial^2 \ln(L)}{\partial l \partial x_0} \Big|_{x_{0,m}, l_m} (x_0 - x_{0,m})(l - l_m) \right]$$

$$Q = \mathbf{X}^T \mathbf{H} \mathbf{X}$$

$\mathbf{H}$  – Hesseova matice

$$\mathbf{X}^T = (x_0 - x_{0,m}, l - l_m)$$

$$\mathbf{H} = \begin{pmatrix} A & C \\ C & B \end{pmatrix} \quad A = \frac{\partial^2 \ln(L)}{\partial x_0^2} \Big|_{x_{0,m}, l_m} \quad B = \frac{\partial^2 \ln(L)}{\partial l^2} \Big|_{x_{0,m}, l_m} \quad C = \frac{\partial^2 \ln(L)}{\partial l \partial x_0} \Big|_{x_{0,m}, l_m}$$

# Problém majáku – odhad neurčitosti

**odhad polohy majáku  $(x_0, l)$**

$$Q = \mathbf{X}^T \mathbf{H} \mathbf{X} \quad \mathbf{H} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$
$$\mathbf{X}^T = (x_0 - x_{0,m}, l - l_m)$$

$$A = \left. \frac{\partial^2 \ln(L)}{\partial x_0^2} \right|_{x_0, m, l_m} = -2 \sum_{i=1}^N \frac{l_m^2 - (x_i - x_{0,m})^2}{[l_m^2 + (x_i - x_{0,m})^2]^2} \quad A < 0$$

$$B = \left. \frac{\partial^2 \ln(L)}{\partial l^2} \right|_{x_0, m, l_m} = -\frac{N}{l_m^2} + 2 \sum_{i=1}^N \frac{l_m^2 - (x_i - x_{0,m})^2}{[l_m^2 + (x_i - x_{0,m})^2]^2} \quad B < 0$$

$$C = \left. \frac{\partial^2 \ln(L)}{\partial l \partial x_0} \right|_{x_0, m, l_m} = -4l_m \sum_{i=1}^N \frac{x_i - x_{0,m}}{[l_m^2 + (x_i - x_{0,m})^2]^2} \quad C^2 < AB$$

# Problém majáku – odhad neurčitosti

$N = 1000$

odhad polohy majáku  $(x_0, l)$

$$Q = \mathbf{X}^T \mathbf{H} \mathbf{X} \quad \mathbf{H} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$

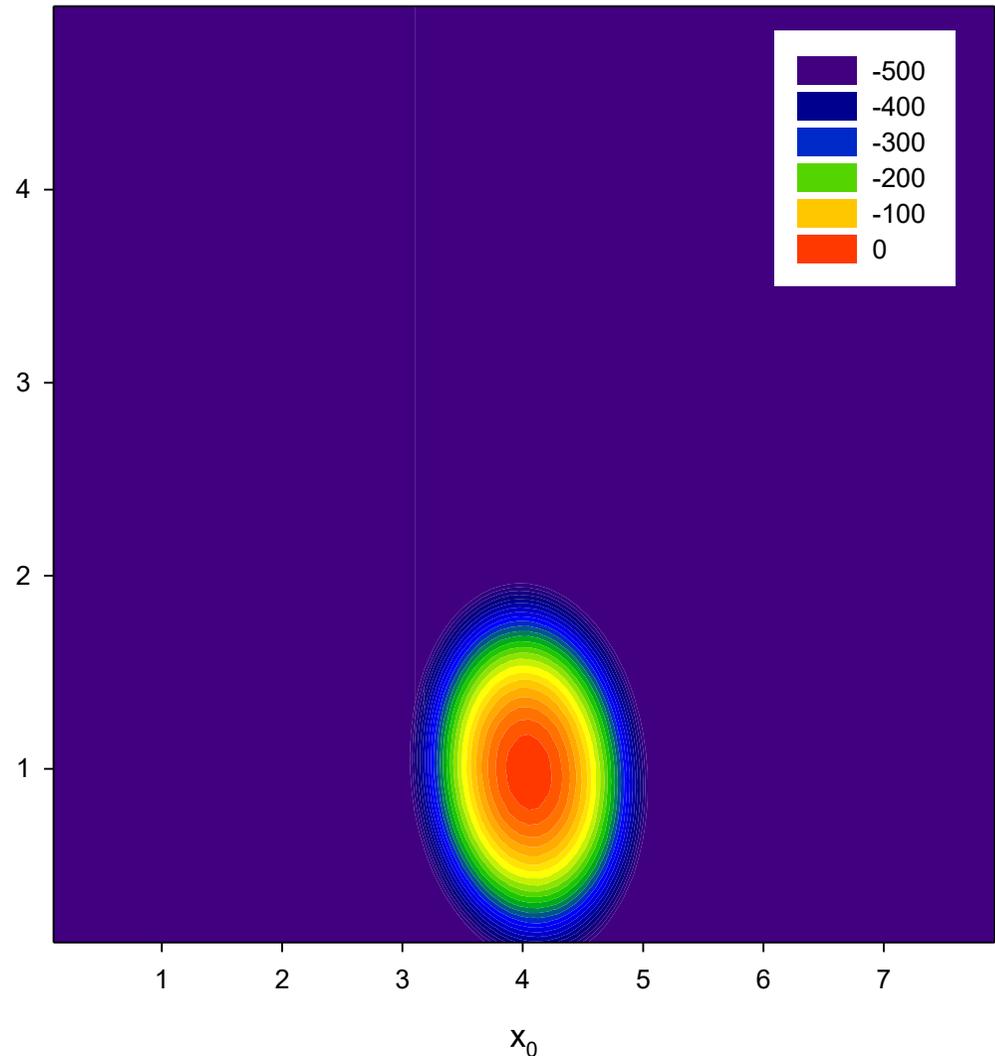
$$\mathbf{X}^T = (x_0 - x_{0,m}, l - l_m)$$

$$f(x_0, l | D_N) \propto L \propto \exp\left(\frac{Q}{2}\right) = \exp\left(\frac{\mathbf{X}^T \mathbf{H} \mathbf{X}}{2}\right)$$

$$2D \text{ Gaussián: } \frac{1}{2\pi\sqrt{\det \mathbf{V}^{-1}}} \exp\left(-\frac{\mathbf{X}^T \mathbf{V}^{-1} \mathbf{X}}{2}\right)$$

$\mathbf{V}$  – kovarianční matice

$$\mathbf{V} = -\mathbf{H}^{-1} = \frac{1}{AB - C^2} \begin{pmatrix} -B & C \\ C & -A \end{pmatrix}$$



# Problém majáku – odhad neurčitosti

$N = 1000$

odhad polohy majáku  $(x_0, l)$

$$Q = \mathbf{X}^T \mathbf{H} \mathbf{X}$$

$$\mathbf{X}^T = (x_0 - x_{0,m}, l - l_m)$$

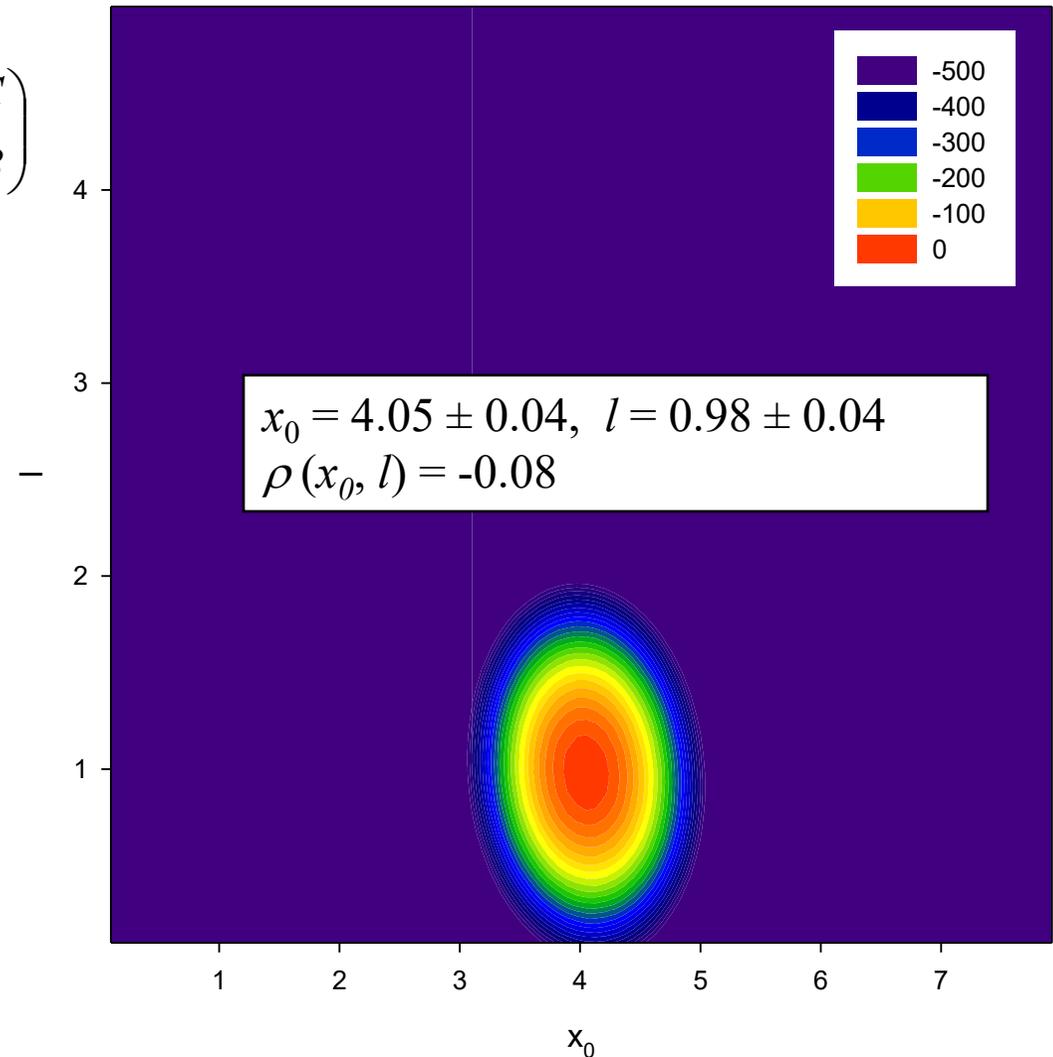
$$\mathbf{H} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$

$$\sigma_{x_0} = \sqrt{\frac{B}{C^2 - AB}}$$

$$\sigma_l = \sqrt{\frac{A}{C^2 - AB}}$$

$$\text{cov}(x_0, l) = \frac{C}{AB - C^2}$$

$$\rho(x_0, l) = \frac{\text{cov}(x_0, l)}{\sigma_{x_0} \sigma_l} = \frac{C}{\sqrt{AB}}$$



# Problém majáku – odhad neurčitosti

$N = 1000$

odhad polohy majáku  $(x_0, l)$

$$Q = \mathbf{X}^T \mathbf{H} \mathbf{X}$$

$$\mathbf{H} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$

$$\mathbf{X}^T = (x_0 - x_{0,m}, l - l_m)$$

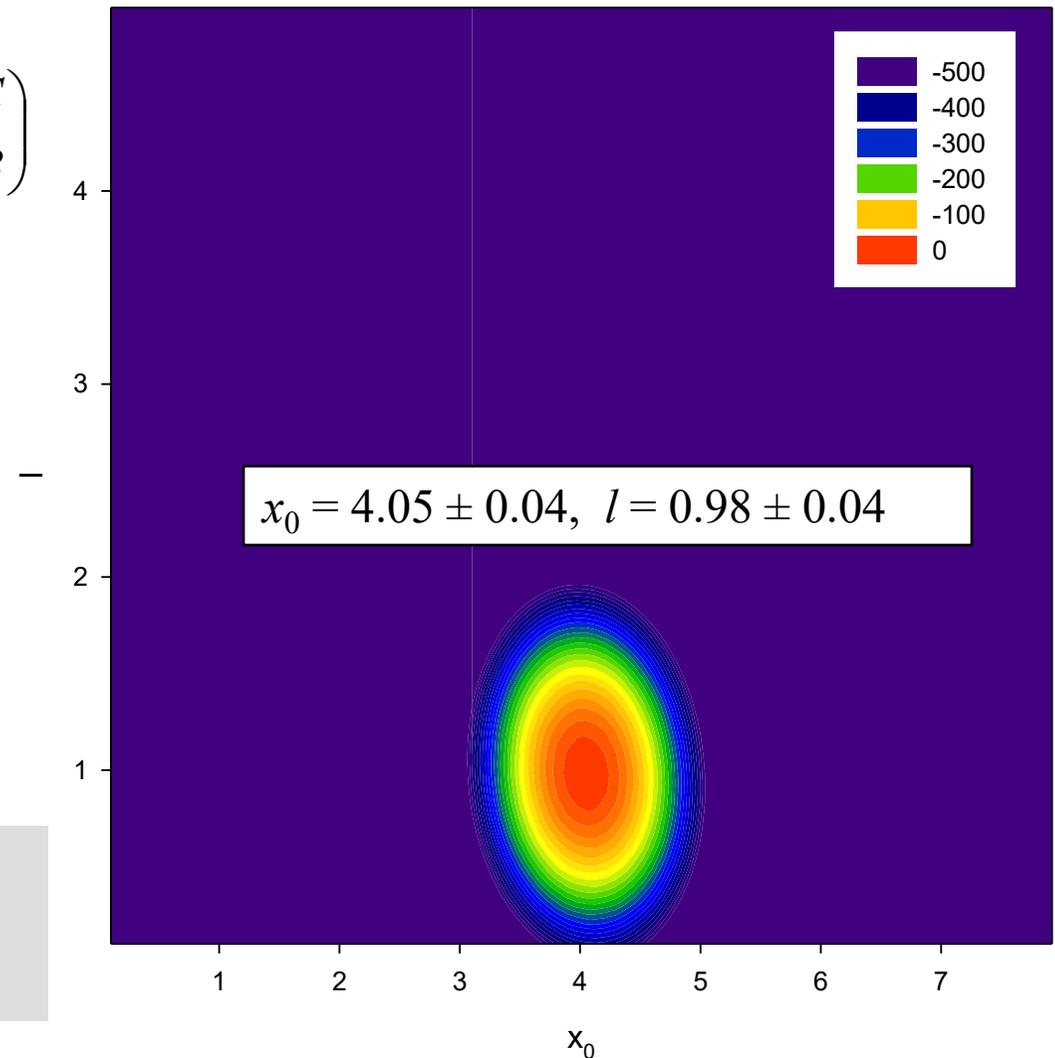
$$f(x_0 | D_N) = \int_{-\infty}^{\infty} f(x_0, l | D_N) dl$$

$$f(x_0, l | D_N) \propto L \propto \exp\left(\frac{Q}{2}\right)$$

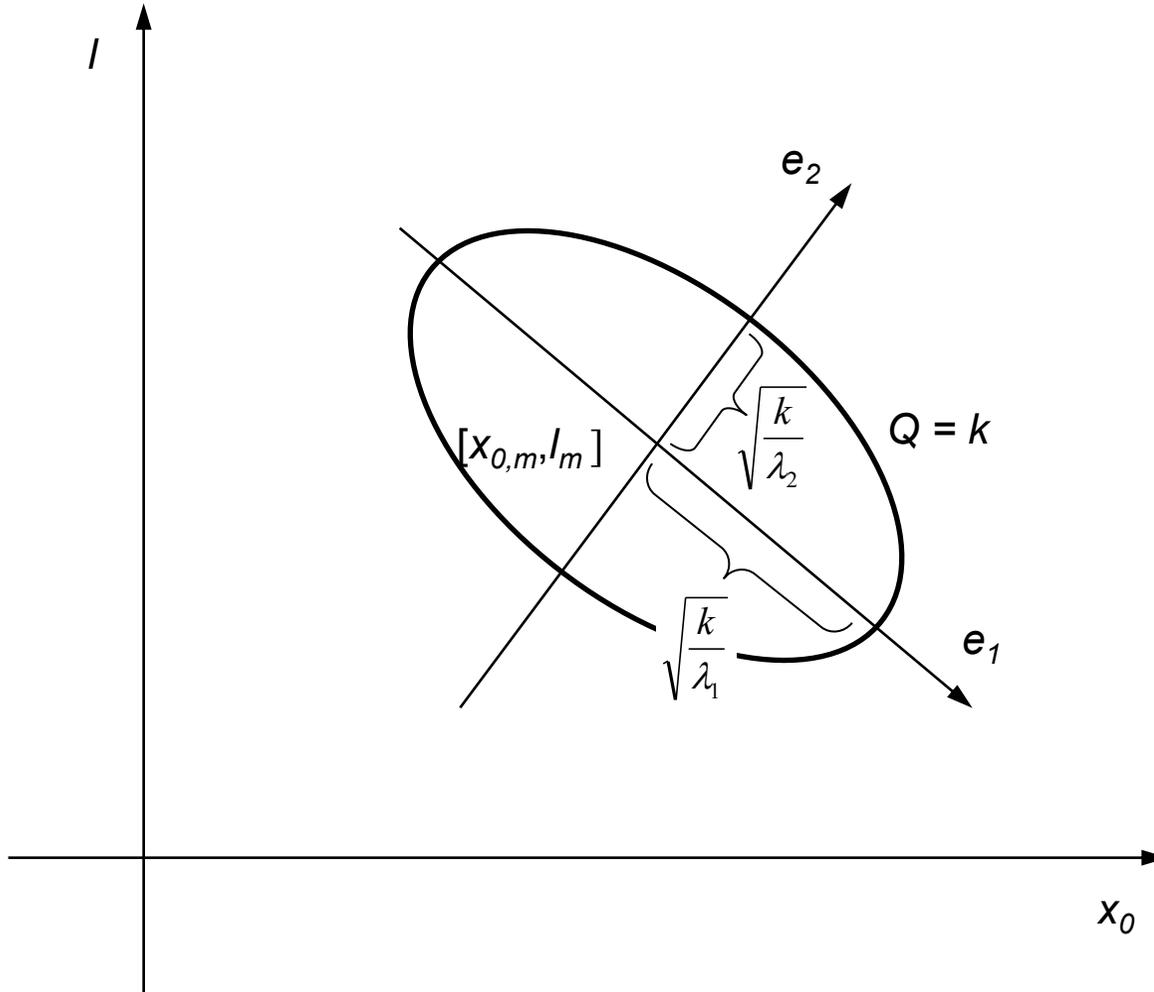
$$f(x_0 | D_N) \propto \exp\left[\frac{1}{2} \frac{AB - C^2}{B} (x_0 - x_{0,m})^2\right]$$

$$\sigma_{x_0} = \sqrt{\frac{B}{C^2 - AB}}$$

$$\sigma_l = \sqrt{\frac{A}{C^2 - AB}}$$



## Dva parametry – odhad neurčitosti



$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} \begin{pmatrix} x_0 \\ l \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ l \end{pmatrix}$$

$$\mathbf{H}\mathbf{X} = \lambda \mathbf{X}$$

$$\mathbf{H}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1$$

$$\mathbf{H}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2$$

# Zobecnění pro $m$ parametrů

- posteriorní hustota pravděpodobnosti

$$f(\mathbf{x}|D_N) \propto L(D_N|\mathbf{x}) \quad \left. \frac{\partial \ln(L)}{\partial x_i} \right|_{\mathbf{x}_0} = 0 \quad i = 1, \dots, m$$

- Taylorův rozvoj  $\ln(L)$ :  $\ln(L) \approx \ln(L(\mathbf{x}_0)) + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \left. \frac{\partial^2 \ln(L)}{\partial x_i \partial x_j} \right|_{\mathbf{x}_0} (x_i - x_{0,i})(x_j - x_{0,j})$

- kvadratická aproximace posteriorní hustoty pravděpodobnosti

$$f(\mathbf{x}|D_N) \propto \exp\left[\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \mathbf{H}(\mathbf{x} - \mathbf{x}_0)\right] \propto \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \mathbf{V}^{-1}(\mathbf{x} - \mathbf{x}_0)\right]$$

$$\mathbf{H} = \nabla^2 \ln[L(\mathbf{x}_0)] \quad H_{i,j} = \left. \frac{\partial^2 \ln(L)}{\partial x_i \partial x_j} \right|_{\mathbf{x}_0} \quad (\textit{m-rozměrný gaussian})$$

- kovarianční matice  $\mathbf{V} = -\mathbf{H}^{-1}$   $V_{i,j} = \text{cov}(x_i, x_j)$

# Zobecnění pro $m$ parametrů

- Taylorův rozvoj funkce  $f(\mathbf{x})$  v bodě  $\mathbf{x}_0$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \mathbf{H}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

$\mathbf{J}(\mathbf{x}_0)$ ..... Jacobiho matice

$$J_i(\mathbf{x}_0) = \left. \frac{\partial f}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0} \quad m \times 1$$

$\mathbf{H}(\mathbf{x}_0)$ ..... Hesseova matice

$$H_{i,j}(\mathbf{x}_0) = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{x}=\mathbf{x}_0} \quad m \times m$$

# Lineární model

- posteriorní hustota pravděpodobnosti

$$f(\mathbf{x}|D_N) = \frac{P(D_N|\mathbf{x})f(\mathbf{x})}{P(D_N)}$$

- pokud je apriorní hustota pravděpodobnosti konstantní:

$$f(\mathbf{x}|D_N) \propto P(D_N|\mathbf{x}) = L(\mathbf{x}) \quad \left. \frac{\partial \ln(L)}{\partial x_i} \right|_{\mathbf{x}_0} = 0 \quad i = 1, \dots, m \quad \longrightarrow \quad \nabla \ln[L(\mathbf{x}_0)] = 0$$

- lineární model:  $\nabla \ln[L(\mathbf{x})] = \mathbf{H}\mathbf{x} + \mathbf{C}$

- odhad parametrů  $\mathbf{x}_0 = -\mathbf{H}^{-1}\mathbf{C}$

$$\nabla^2 \ln(L) = \mathbf{H}$$

- neurčitost odhadu

$$V_{i,j} = \text{cov}(x_i, x_j) \quad \mathbf{V} = -\mathbf{H}^{-1}$$

# Iterativní linearizace

- Taylorův rozvoj  $\ln L$  v bodě  $\mathbf{x}_1$

$$\ln[L(\mathbf{x})] \approx \ln[L(\mathbf{x}_1)] + \underbrace{(\mathbf{x} - \mathbf{x}_1)^T \nabla \ln[L(\mathbf{x}_1)]}_{m \times 1} + \frac{1}{2} (\mathbf{x} - \mathbf{x}_1)^T \underbrace{\nabla^2 \ln[L(\mathbf{x}_1)]}_{m \times m} (\mathbf{x} - \mathbf{x}_1)$$

$$\nabla \ln[L(\mathbf{x})] \approx \nabla \ln[L(\mathbf{x}_1)] + \nabla^2 \ln[L(\mathbf{x}_1)](\mathbf{x} - \mathbf{x}_1)$$

- pro bod  $\mathbf{x}_0$ , kde  $L$  nabývá maxima:

$$\mathbf{x}_0 \approx \mathbf{x}_1 - (\nabla^2 \ln[L(\mathbf{x}_1)])^{-1} \nabla \ln[L(\mathbf{x}_1)]$$

- platí přesně pokud  $\mathbf{x}_1 = \mathbf{x}_0$
- nebo  $\ln L$  je lineární funkce

## Newton-Raphsonův algoritmus

1. zvol počáteční odhad parametrů  $\mathbf{x}_1$

2. Vypočítej  $\nabla \ln[L(\mathbf{x}_1)]$ ,  $\nabla^2 \ln[L(\mathbf{x}_1)]$  ← polož  $\mathbf{x}_1 = \mathbf{x}_2$ , opakuj dokud  $\nabla \ln[L(\mathbf{x}_1)] = 0$

3. Vypočítej upřesněný odhad  $\mathbf{x}_2 \approx \mathbf{x}_1 - (\nabla^2 \ln[L(\mathbf{x}_1)])^{-1} \nabla \ln[L(\mathbf{x}_1)]$

# Iterativní linearizace

- Newton-Raphsonův algoritmus

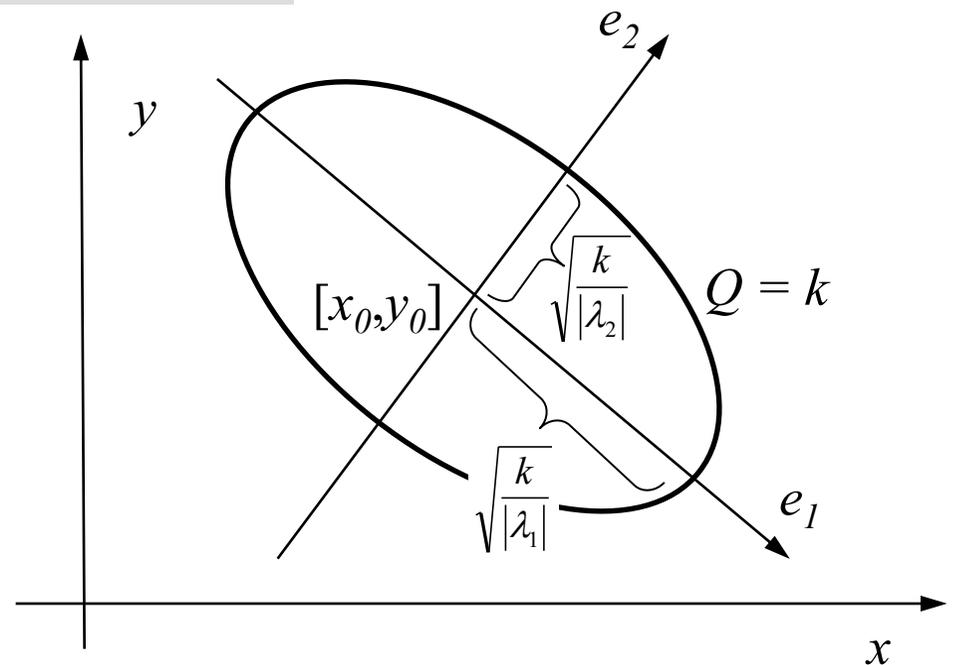
$$\mathbf{x}_2 \approx \mathbf{x}_1 - (\nabla^2 \ln[L(\mathbf{x}_1)])^{-1} \nabla \ln[L(\mathbf{x}_1)]$$

- zlepšení stability ( $c < 0$  malé,  $\mathbf{E}$  – jednotková matice)

$$\mathbf{x}_2 \approx \mathbf{x}_1 - (\nabla^2 \ln[L(\mathbf{x}_1)] + c\mathbf{E})^{-1} \nabla \ln[L(\mathbf{x}_1)]$$

$$\nabla^2 \ln[L(\mathbf{x}_1)] e_i = \lambda_i e_i \quad \lambda_i < 0$$

$$(\nabla^2 \ln[L(\mathbf{x}_1)] + c\mathbf{E}) e_i = (\lambda_i + c) e_i$$



# Odhad parametrů pro normální rozdělení

- parametry  $\mu, \sigma$  ( $m = 2$ )
- posteriorní hustota pravděpodobnosti:  $f(\mu, \sigma | D_N) \propto P(D_N | \mu, \sigma) f(\mu, \sigma)$
- apriorní hustota pravděpodobnosti:  $f(\mu, \sigma) \propto \text{konst.}$

$$f(\mu, \sigma | D_N) = \frac{1}{(2\pi)^{\frac{N}{2}}} \frac{1}{\sigma^N} \exp \left[ - \sum_{i=1}^N \frac{(x_i - \mu)^2}{2\sigma^2} \right]$$

$$\ln f = - \sum_{i=1}^N \frac{(x_i - \mu)^2}{2\sigma^2} - N \ln(\sigma) - N \ln(\sqrt{2\pi})$$

$$\left. \frac{\partial \ln(L)}{\partial \mu} \right|_{\mu_0, \sigma_0} = 0 \quad \longrightarrow \quad \sum_{i=1}^N \frac{x_i - \mu_0}{\sigma_0^2} = 0 \quad \longrightarrow \quad \mu_0 = \frac{1}{N} \sum_{i=1}^N x_i \equiv \bar{x}$$

$$\left. \frac{\partial \ln(L)}{\partial \sigma} \right|_{\mu_0, \sigma_0} = 0 \quad \longrightarrow \quad \sum_{i=1}^N \frac{(x_i - \mu_0)^2}{\sigma_0^3} - \frac{N}{\sigma_0} = 0 \quad \longrightarrow \quad \sigma_0^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2$$