Large-order analysis of the convergent renormalized strong-coupling perturbation theory for the quartic anharmonic oscillator

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Two hundred coefficients of the renormalized strong-coupling perturbation expansion for the ground and first excited states of the quartic anharmonic oscillator are calculated numerically. The large-order behavior of the perturbation coefficients is analyzed, a general and comparatively simple analytic formula describing their large-order behavior is proposed, and it is shown that this formula is consistent with known results from the divergent weak-coupling expansion. The accuracy of our numerically determined coefficients is checked by summation rules. In particular, if the summation rules are supplemented by the leading terms of our large-order formula, we obtain remarkably accurate results. This independently confirms the correctness of our large-order analysis. It is shown that the renormalized strong-coupling expansion converges—in contrast to other perturbation expansions—for all physically relevant coupling constants. [S1050-2947(97)03712-8]

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I. INTRODUCTION

We investigate the Schrödinger equation $H(\beta)\psi = E(\beta)\psi$ for the quartic anharmonic oscillator, where

$$H(\beta) = p^2 + x^2 + \beta x^4, \quad \beta \geq 0. \tag{1}$$

This is one of the old, but nontrivial problems of quantum mechanics. As is well known, $E(\beta)$ can be expressed as a weak-coupling perturbation series in powers of β, which diverges for every $\beta > 0$ [1–4]. Hamiltonian (1) can be transformed into an equivalent Hamiltonian $H = \beta^{1/3}[p^2 + \beta^{-2/3}x^2 + x^4]$ [3]. Consequently, $E(\beta)$ also possesses the strong-coupling expansion

$$E(\beta) = \beta^{1/3} \sum_{n=0}^{\infty} K_n \beta^{-2n/3}. \tag{2}$$

This series converges if β is large [3,4]. Unfortunately, the perturbative computation of the coefficients K_n is very difficult [5–9].

An alternative perturbative approach based upon renormalization (Wick ordering [10] or scaling [9–14]) has considerable conceptual and technical advantages. In the quartic case, Wick ordering and scaling are closely related, and they differ by a numerical factor in the effective coupling constant. In the scaling approach, $\beta \in [0,\infty)$ is replaced by a renormalized coupling constant $\kappa \in [0,1)$ according to $\beta = \kappa/[3(1-\kappa)^{3/2}]$, and Hamiltonian (1) is transformed into a renormalized Hamiltonian $H_R(\kappa)$ [11,12]:

$$H(\beta) = (1-\kappa)^{-1/2} H_R(\kappa), \tag{3}$$

$$H_R(\kappa) = p^2 + x^4/3 + (1-\kappa)(x^2 - x^4/3), \tag{4}$$

$$E(\beta) = (1-\kappa)^{-1/2} E_R(\kappa). \tag{5}$$

In contrast to Eq. (1), the perturbation in $H_R(\kappa)$ is a difference of two terms, which partly compensate for each other [11,12]. The renormalized energy $E_R(\kappa)$ can either be expressed as a divergent weak-coupling expansion in κ [12], or as a strong-coupling expansion in $1-\kappa$ [14],

$$E_R(\kappa) = \sum_{n=0}^{\infty} c_n \kappa^n = \sum_{n=0}^{\infty} \Gamma_n (1-\kappa)^n. \tag{6}$$

The advantage of the renormalized approach is due to the fact that $E_R(\kappa)$ is finite for $\kappa \in [0,1]$ [$E_R(0) = 1$ and $E_R(1) = \Gamma_0$], since the troublesome pole $(1-\kappa)^{-1/2}$ is explicitly factorized out in Eq. (5).

The weak-coupling expansion for $E_R(\kappa)$ diverges almost as strongly as the corresponding weak-coupling expansion for $E(\beta)$ [10,12]. In contrast, it was shown in theorems 1 and 2 of Ref. [14] that the strong-coupling expansion for $E_R(\kappa)$ is analytic at $\kappa = 1$, which implies that it converges if κ is close to 1. Moreover, Table V in Ref. [14] indicates that this strong-coupling expansion actually converges for all physically relevant $\kappa \in [0,1)$.

The main purpose of this paper is to study the large-order behavior of the perturbation series coefficients in the strong-coupling case. We show that this large-order behavior is exceptionally simple in the renormalized case. This provides us with an interesting insight which can be used even for the study of the strong-coupling expansion (2). In contrast to a large-order analysis of divergent expansions, our large-order analysis can be used directly for numerical purposes.

II. NUMERICAL CALCULATIONS

In this paper, we compute numerically 200 coefficients Γ_n for the ground and first excited states of the quartic anharmonic oscillator, perform their large-order analysis, and propose an analytic large-order formula for Γ_n. With the help of this formula, we show that the strong-coupling expansion for

$E_R(\kappa)$ converges for all physically relevant $\kappa \in [0,1)$.

Such a large-order analysis could not be done before, since only a comparatively small number of coefficients Γ_n could be computed [14]. Here we use a method [15] which is able to produce very accurate results at comparatively low computational costs even for very large perturbation orders, and which can also be used for the direct computation of the coefficients K_n in Eq. (2) [16].

We assume that an eigenfunction ψ of the renormalized Hamiltonian H_R possesses a strong-coupling expansion

$$\psi = \sum_{n=0}^{\infty} \psi_n (1-\kappa)^n. \tag{7}$$

Standard perturbative approach leads to the equations

$$\left(p^2 + \frac{x^4}{3}\right)\psi_0 = \Gamma_0 \psi_0, \tag{8}$$

$$\left(p^2 + \frac{x^4}{3}\right)\psi_n + \left(x^2 - \frac{x^4}{3}\right)\psi_{n-1} = \sum_{j=0}^{n} \Gamma_j \psi_{n-j} \tag{9}$$

for $n=0$ and $n \geq 1$, respectively. For the technique of the solution of Eqs. (8) and (9) we refer to Ref. [15]. This method is combined with the Taylor expansion of the wave functions. As indicated in Ref. [15], we replace the boundary condition at infinity by a boundary condition at a sufficiently large number x_0. Our results depend on x_0, and the highest power x^N occurring in the Taylor expansion. However, we obtain very accurate results if x_0 and N are sufficiently large.

As a test, we calculated the energies of the ground and first excited state of the quartic oscillator with Hamiltonian $H = p^2 + x^4$ in MAPLE, using 100 decimal digits, $N=1300$, and $x_0 = 6.5$. In this way, we obtained an accuracy of at least 75 decimal digits, which is much better than previous results [6,7].

We also calculated 200 coefficients Γ_n for the ground and first excited states in MAPLE, using 75 decimal digits, $N=2000$, and $x_0 = 8$. In Table I, only some selected coefficients are shown. For $n \geq 2$, all Γ_n are negative.

If the Γ_n are known, the coefficients K_n of the strong-coupling expansion (2) can be computed by using either Eq. (13) of Ref. [9] or standard series manipulation techniques of MAPLE. In the latter way, we calculated 100 coefficients K_n for the ground and first excited state. So far, only relatively few K_n could be computed perturbatively [5–9]. Unfortunately, a large-order analysis of the K_n seems to be difficult since their signs change quite irregularly. Therefore, we restricted our attention to the Γ_n.

The accuracy of our coefficients Γ_n can be checked by summation rules. The first obvious rule follows from the fact that Hamiltonian (1) describes for $\beta = 0$ the harmonic oscillator with energies $E(0) = 2K+1$, $K = 0,1,2,\ldots$. This implies [14]

$$\Sigma_0 = \sum_{n=0}^{\infty} \Gamma_n = E(0) = 2K+1. \tag{10}$$

Further summation rules can be derived by calculating the derivatives of both the weak- and strong-coupling expansion in Eq. (6) with respect to κ. Setting $\kappa = 0$ then yields

$$\Sigma_j = \sum_{n=0}^{\infty} [n(n-1)\cdots(n-j+1)\Gamma_n] = (-1)^j j! c_j. \tag{11}$$

MAPLE programs, which compute the ground state coefficients c_j exactly, are described in Refs. [12,13].

For our numerically determined coefficients Γ_n, we compute the partial sums

$$\Sigma_j^{(N)} = \sum_{n=0}^{N} [n(n-1)\cdots(n-j+1)\Gamma_n], \quad j \geq 1. \tag{12}$$

For $N=200$, the exact Σ_j as well as the differences $\Sigma_j^{(N)} - \Sigma_j$ are shown in Table II. For $j=0$, 1, and 2, the differences $\Sigma_j^{(N)} - \Sigma_j$ are very small, and show the high accuracy of our numerical results. For $j=3$, 4, and 5, they become larger because of the increasing weight of the coefficients Γ_n with $n > N$. Moreover, convergence is apparently slower for the first excited state than for the ground state. We also note that $\Sigma_j^{(N)} > \Sigma_j$, which is a consequence of the fact that for $n \geq 2$ all Γ_n are negative.

III. LARGE-ORDER BEHAVIOR OF Γ_n COEFFICIENTS

In order to study the large-order behavior of Γ_n, we investigate the ratio Γ_n/Γ_{n-1} which occurs in the d'Alembert convergence criterion. This ratio was extrapolated using the Richardson scheme [17] in the variable $1/n^{1/2}$. For the ground state, we found that the large-order behavior of this ratio can be described by the following truncated expansion in $1/n^{1/2}$:

$$\Gamma_n^{(0)}/\Gamma_{n-1}^{(0)} = 1 - \frac{2^{1/2}}{n^{1/2}} + \frac{1}{2n}. \tag{13}$$

The values of the coefficients in Eq. (13) depend on the interval of indices n of Γ_n which were used in the interpolation. We tried interpolation intervals of different lengths in the range $181 \leq n \leq 200$. The values of the first two coefficients in Eq. (13) are not very sensitive to the interval being used. However, the value of the third coefficient is less accurate and oscillates around $\frac{1}{2}$. Nevertheless, we are confident that also this coefficient is correct. The large-order formula for $\Gamma_n^{(0)}$, which is consistent with Eq. (13), has the form

$$\Gamma_n^{(0)} = A^{(0)} \frac{e^{-2\sqrt{2n}}}{\sqrt{2n}}. \tag{14}$$

Here $A^{(0)}$ is a constant.

Analogous calculations for the first excited state showed that $\Gamma_n^{(1)}/\Gamma_{n-1}^{(1)}$ can be described by

TABLE I. Selected perturbation coefficients Γ_n of the strong-coupling expansion for the ground and first excited state energy $E_R(\kappa)$ of the renormalized quartic oscillator.

n	Ground state $\Gamma_n^{(0)}$	First excited state $\Gamma_n^{(1)}$
0	0.735 214 010 331 216	2.634 546 134 058 831
1	0.277 055 672 879 946	0.422 158 671 146 023
2	$-0.111\,788\,972\,096\,450\times10^{-1}$	$-0.317\,297\,121\,676\,530\times10^{-1}$
3	$-0.466\,149\,311\,582\,119\times10^{-3}$	$-0.118\,348\,533\,354\,816\times10^{-1}$
4	$-0.293\,444\,235\,328\,683\times10^{-3}$	$-0.566\,020\,610\,884\,992\times10^{-2}$
5	$-0.148\,065\,256\,807\,374\times10^{-3}$	$-0.298\,444\,332\,180\,920\times10^{-2}$
6	$-0.769\,154\,429\,963\,378\times10^{-4}$	$-0.168\,196\,037\,910\,155\times10^{-2}$
7	$-0.423\,206\,186\,488\,083\times10^{-4}$	$-0.996\,162\,298\,547\,618\times10^{-3}$
8	$-0.242\,857\,940\,621\,442\times10^{-4}$	$-0.613\,183\,501\,076\,960\times10^{-3}$
9	$-0.144\,250\,579\,507\,101\times10^{-4}$	$-0.389\,299\,010\,839\,413\times10^{-3}$
10	$-0.882\,035\,796\,048\,397\times10^{-5}$	$-0.253\,531\,925\,098\,163\times10^{-3}$
11	$-0.552\,867\,849\,653\,391\times10^{-5}$	$-0.168\,684\,451\,476\,763\times10^{-3}$
12	$-0.354\,065\,902\,908\,941\times10^{-5}$	$-0.114\,304\,581\,919\,298\times10^{-3}$
13	$-0.231\,060\,058\,334\,935\times10^{-5}$	$-0.786\,945\,770\,110\,203\times10^{-4}$
14	$-0.153\,323\,581\,119\,562\times10^{-5}$	$-0.549\,385\,838\,893\,818\times10^{-4}$
15	$-0.103\,267\,824\,141\,986\times10^{-5}$	$-0.388\,309\,856\,406\,983\times10^{-4}$
16	$-0.704\,933\,716\,265\,410\times10^{-6}$	$-0.277\,513\,646\,802\,810\times10^{-4}$
17	$-0.487\,094\,742\,781\,166\times10^{-6}$	$-0.200\,320\,156\,542\,175\times10^{-4}$
18	$-0.340\,325\,648\,653\,583\times10^{-6}$	$-0.145\,915\,572\,269\,964\times10^{-4}$
19	$-0.240\,209\,641\,371\,477\times10^{-6}$	$-0.107\,170\,633\,904\,950\times10^{-4}$
20	$-0.171\,140\,073\,870\,888\times10^{-6}$	$-0.793\,148\,696\,590\,617\times10^{-5}$
21	$-0.122\,990\,906\,347\,772\times10^{-6}$	$-0.591\,131\,390\,412\,770\times10^{-5}$
22	$-0.891\,015\,087\,373\,825\times10^{-7}$	$-0.443\,446\,990\,479\,762\times10^{-5}$
23	$-0.650\,353\,066\,160\,676\times10^{-7}$	$-0.334\,681\,619\,010\,113\times10^{-5}$
24	$-0.478\,027\,227\,703\,690\times10^{-7}$	$-0.254\,027\,612\,001\,870\times10^{-5}$
25	$-0.353\,675\,330\,095\,645\times10^{-7}$	$-0.193\,835\,914\,598\,832\times10^{-5}$
26	$-0.263\,289\,801\,622\,188\times10^{-7}$	$-0.148\,646\,023\,997\,306\times10^{-5}$
27	$-0.197\,144\,828\,033\,614\times10^{-7}$	$-0.114\,528\,455\,544\,143\times10^{-5}$
28	$-0.148\,428\,783\,780\,867\times10^{-7}$	$-0.886\,343\,135\,165\,537\times10^{-6}$
29	$-0.112\,332\,167\,365\,367\times10^{-7}$	$-0.688\,837\,022\,415\,338\times10^{-6}$
30	$-0.854\,331\,361\,307\,300\times10^{-8}$	$-0.537\,482\,145\,539\,263\times10^{-6}$
40	$-0.692\,427\,651\,200\,224\times10^{-9}$	$-0.539\,973\,504\,236\,571\times10^{-7}$
50	$-0.763\,555\,251\,883\,617\times10^{-10}$	$-0.699\,402\,643\,129\,441\times10^{-8}$
60	$-0.104\,682\,599\,734\,451\times10^{-10}$	$-0.108\,816\,065\,189\,052\times10^{-8}$
70	$-0.169\,163\,425\,861\,465\times10^{-11}$	$-0.194\,948\,682\,414\,633\times10^{-9}$
80	$-0.311\,233\,024\,257\,551\times10^{-12}$	$-0.391\,114\,882\,919\,387\times10^{-10}$
90	$-0.636\,481\,899\,357\,226\times10^{-13}$	$-0.861\,610\,800\,449\,058\times10^{-11}$
100	$-0.142\,178\,964\,205\,299\times10^{-13}$	$-0.205\,416\,058\,730\,723\times10^{-11}$
125	$-0.455\,898\,856\,814\,737\times10^{-15}$	$-0.753\,137\,381\,819\,770\times10^{-13}$
150	$-0.204\,948\,564\,682\,983\times10^{-16}$	$-0.376\,559\,140\,065\,207\times10^{-14}$
175	$-0.118\,903\,133\,815\,519\times10^{-17}$	$-0.238\,602\,260\,079\,994\times10^{-15}$
200	$-0.843\,663\,366\,544\times10^{-19}$	$-0.182\,543\,904\,091\times10^{-16}$

$$\Gamma_n^{(1)}/\Gamma_{n-1}^{(1)} = 1 - \frac{2^{1/2}}{n^{1/2}} + \frac{1}{n}. \tag{15}$$

This leads to the large-order formula

$$\Gamma_n^{(1)} = A^{(1)} e^{-2\sqrt{2n}}, \tag{16}$$

where $A^{(1)}$ is a constant.

On the basis of Eqs. (14) and (16), we conjecture that the large-order expansion for the $\Gamma_n^{(K)}$ has the form

$$\Gamma_n^{(K)} = A^{(K)} (2n)^{(K-1)/2} e^{-2\sqrt{2n}} \left(1 + \sum_{m=1}^{\infty} \frac{a_m^{(K)}}{(2n)^{m/2}} \right). \tag{17}$$

Here, $K=0$ corresponds to the ground state, and $K=1,2,\ldots$ correspond to excited states.

Next we discuss the analytical calculation of the coefficients $A^{(K)}$ and $a_m^{(K)}$. There are two possibilities: The first one is to transform the known results for the renormalized weak-coupling case. The second one is to consider tunnelling

TABLE II. Summation rules for the coefficients Γ_n of the strong-coupling expansion for the ground and first excited state energy $E_R(\kappa)$ of the quartic anharmonic oscillator. Σ_j is the exact value of the summation rule for the infinite number of terms. $\Sigma_j^{(N)}$ denotes the partial sum for $N=200$. $\Sigma_j^{\mathrm{LO}nu}$ ($\Sigma_j^{\mathrm{LO}an}$) denotes the partial sum $\Sigma_j^{(N)}$ plus the remaining part of the series in which the truncated large-order formula (17) with the numerical estimates (analytical values) of the $a_m^{(K)}$ coefficients is used. This part of the series was calculated by extending the upper limit in the sum to 1500.

	Ground state				First excited state		
j	Σ_j	$\Sigma_j^{(N)}-\Sigma_j$	$\Sigma_j^{\mathrm{LO}nu}-\Sigma_j$	$\Sigma_j^{\mathrm{LO}an}-\Sigma_j$	Σ_j	$\Sigma_j^{(N)}-\Sigma_j$	$\Sigma_j^{\mathrm{LO}nu}-\Sigma_j$
0	1	0.803×10^{-18}	-0.216×10^{-21}	-0.133×10^{-22}	3	0.178×10^{-15}	-0.412×10^{-18}
1	0.25	0.169×10^{-15}	-0.454×10^{-19}	-0.279×10^{-20}	0.25	0.377×10^{-13}	-0.865×10^{-16}
2	$-0.041\overline{6}$	0.356×10^{-13}	-0.951×10^{-17}	-0.584×10^{-18}	$-0.541\overline{6}$	0.795×10^{-11}	-0.181×10^{-13}
3	$-0.093\ 75$	0.747×10^{-11}	-0.199×10^{-14}	-0.121×10^{-15}	$-2.781\ 25$	0.167×10^{-8}	-0.379×10^{-11}
4	$-0.686\ 631\ 94\overline{4}$	0.156×10^{-8}	-0.416×10^{-12}	-0.252×10^{-13}	$-26.228\ 298\ 61\overline{}$	0.351×10^{-6}	-0.790×10^{-9}
5	$-7.891\ 710\ 069\ \overline{4}$	0.327×10^{-6}	-0.867×10^{-10}	-0.524×10^{-11}	$-383.510\ 199\ 652\ \overline{7}$	0.736×10^{-4}	-0.165×10^{-6}

through a peak given by a negative harmonic term combined with a positive quartic term. The latter possibility would have the advantage of certain visualization of the problem. However, in this case we would be obliged to start from the beginning, while in the case of the first technique we can use the renormalized weak-coupling results needed for the calculation of the $A^{(K)}$ and $a_m^{(K)}$ coefficients. We shall see that even with this advantage the calculation of the coefficients is a nontrivial problem. We shall return to the second technique in a forthcoming paper.

For the above purpose we use the summation rule (11), and assume that j is large. If we replace summation by integration, we have to calculate integrals of the form

$$I_m^{(K)}=\int_j^\infty \frac{x!}{(x-j)!}\,\frac{\exp(-2\sqrt{2x})}{(2x)^{(m+1-K)/2}}\,dx. \tag{18}$$

Here, we use the convention $x!=\Gamma(x+1)$ also for nonintegral x. The leading term of this integral for large j is

$$I_m^{(K)}=\frac{(j^2/2)!}{(j^2/2-j)!}\,\frac{e^{-j}}{j^{m-K}}\sqrt{\pi j^3}. \tag{19}$$

Further, it can be shown that

$$\frac{I_m^{(K)}2^j}{j!(j+K-1/2)!}=\frac{1}{2e\pi^{1/2}j^m}\left(1+\sum_{l=1}^{\infty}\frac{d_l}{j^l}\right). \tag{20}$$

Here, the d_l are constants. In the next step, we insert this equation and the large-order formula for $c_j^{(K)}$ [10,12]

$$c_j^{(K)}=\frac{(-1)^{j+1}12^K24^{1/2}(j+K-1/2)!}{2^jK!\,\pi^{3/2}e^3}\left(1+\sum_{m=1}^{\infty}\frac{f_m}{j^m}\right), \tag{21}$$

where the f_m are constants, into Eq. (11). In this way, it can be shown that Eq. (17) is consistent with the results given in Ref. [10] for all orders of $1/\sqrt{2n}$. This indicates that our ansatz (17) is justified. Since Eq. (21) is of semiclassical character, the large-order behavior in Eq. (17) is also of semiclassical character. Here, semiclassical character stands for the JWKB approximation supplemented by higher-order terms.

First, we analytically calculated the coefficient $A^{(K)}$ via Eqs. (11), (17) and (18), yielding

$$A^{(K)}=\lim_{j\to\infty}\frac{(-1)^j j!\,c_j^{(K)}}{I_0^{(K)}}=-\frac{12^K}{K!}\,\frac{4\sqrt{6}}{\pi e^2}. \tag{22}$$

With the help of Richardson extrapolation, we estimated the higher-order coefficients $a_m^{(K)}$ in Eq. (17) from the numerical coefficients Γ_n. For the ground state, we obtained $a_1^{(0)}=-1.15$ and $a_2^{(0)}=-0.5$. The truncated expression (17) with these coefficients is a good approximation to the actual values of the coefficients $\Gamma_n^{(0)}$. Starting from $n=85$, its relative accuracy is better than 10^{-3}. Now we can calculate the infinite series in the summation rules in such a way that we use the numerical values of the coefficients $\Gamma_n^{(K)}$ for $0\leq n\leq N$ and the large-order formula (17) for $n>N$. It follows from Table II that the use of the truncated large-order expression (17) improves the accuracy of the summation rules for $j=0,\ldots,5$ by 3–4 orders, which independently confirms the correctness of our large-order analysis. The error of the summation rules supplemented by our asymptotic results lies in the range from 10^{-21} to 10^{-10}. This shows that our numerically calculated coefficients $\Gamma_n^{(0)}$ with $0\leq n\leq 200$ supplemented by the truncated large-order formula (17) for $n>200$ provide extremely accurate results.

For the first excited state, we obtained only one coefficient $a_1^{(1)}=-2.99$. Starting from $n=108$, the relative accuracy of the truncated expression (17) is better than 10^{-3}. We see from Table II that the use of the truncated large order expression (17) for $n>200$ improves the summation rules by 2–3 orders. Again, the summation rules are obeyed with remarkable accuracy.

Finally, we calculated four coefficients $a_m^{(0)}$ analytically. These quantities were calculated from the coefficients $f_1,\ldots,f_4$ in Eq. (21). The coefficient f_1 is taken from the results of Ref. [10], f_2, f_3, and f_4, were calculated with some effort using the results of Ref. [18]. We obtained $a_1^{(0)}=-83/72$, $a_2^{(0)}=-5243/10\,368$, $a_3^{(0)}=-5\,949\,823/$

11 197 440, and $a_4^{(0)} = -1\,526\,347\,139/3\,224\,862\,720$ which are in good agreement with estimated values for $a_1^{(0)}$ and $a_2^{(0)}$ given above. Using these coefficients it is seen that Eq. (17) is qualitatively correct already for $n=4$, while the relative accuracy for $n=25$ ($n=200$) is 3×10^{-3} (2×10^{-5}), respectively. If these coefficients are used in the summation rules, the accuracy of the differences $\Sigma_j^{\mathrm{LO}}-\Sigma_j$ improves considerably, as seen in Table II. Detailed description of this calculation as well as the calculation of the coefficients $a_m^{(K)}$ for the excited states will be published separately.

IV. CONCLUSIONS

The results of this paper may be summarized as follows. Using 200 numerically calculated perturbation coefficients Γ_n for the ground and first excited state energy of the quartic anharmonic oscillator, we investigated the large-order behavior of the renormalized strong coupling expansion for $E_R(\kappa)$. We showed that the perturbation coefficients Γ_n permit—unlike the coefficients K_n of the strong-coupling expansion (2)—a relatively easy large order analysis, and found that the coefficients Γ_n can be described by the analytic large order formula (17), which is consistent with known results from the divergent weak-coupling expansion [10]. In this paper, the leading term $A^{(K)}$ and the coefficients $a_1^{(0)}$, ..., $a_4^{(0)}$ were calculated analytically. Further analytic coefficients $a_m^{(K)}$ can be calculated using Eq. (21) and results from Refs. [10,18]. The coefficient $a_1^{(1)}$ was estimated numerically. The summation rules (10) and (11) for Γ_n are obeyed with remarkable accuracy and show that our numerically computed coefficients Γ_n supplemented by the truncated large-order formula (17) are apparently very close to the exact ones. The convergence of the strong-coupling expansion for $E_R(\kappa)$ for all $\kappa\in[0,1]$ follows from the large-order formula (17). Expansion (17) is expected to be only asymptotic. However, the absolute value of the leading term in Eq. (17) is an upper bound to the absolute value of Γ_n (see the negative signs of the differences $\Sigma_j^{\mathrm{LO}}-\Sigma_j$ in Table II). Thus the strong-coupling expansion for $E_R(\kappa)$ converges for all $\kappa\in[0,1]$, and the energy $E(\beta)$ of the quartic anharmonic oscillator can for all physically relevant coupling constants $\beta\in[0,\infty)$ be computed by the *convergent* renormalized strong-coupling expansion.

So far, perturbative calculation of the energy eigenvalues of the anharmonic oscillators involved strongly divergent perturbation series (the standard and renormalized weak-coupling cases) or the series converging for sufficiently large β (the standard strong-coupling case). The results of this paper show that these difficulties can be avoided if the renormalized strong-coupling perturbation series is used. We showed that the perturbation theory is convergent in this case for all the physical values of the coupling constant β, and that the large-order behavior of the perturbation coefficients can be described by a simple analytical formula. A natural question is what is the physical content of the large-order behavior described by Eq. (17). It follows from Sec. III that the large-order behavior of both the strong- and weak-coupling expansions of $E_R(\kappa)$ is of semiclassical character. However, the strong-coupling expansion is convergent and structurally more simple. From this point of view, the renormalized strong-coupling expansion is the most natural perturbative approach for the anharmonic oscillators.

The quartic anharmonic oscillator is a very important model problem in quantum mechanics and quantum field theory, and, consequently, the conclusions given above are of considerable significance. We hope to do similar investigations not only for various anharmonic oscillators and the hydrogen atom in a magnetic field, but also in quantum field theory. Concluding, we would like to state that from the mathematical point of view it would be highly desirable to put our results into a completely rigorous form in the spirit of Ref. [19].

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