Method for calculating analytical solutions of the Schrödinger equation: Anharmonic oscillators and generalized Morse oscillators

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A method for calculating the analytical solutions of the one-dimensional Schrödinger equation is suggested. A general discussion of the possible forms of the potentials and wave functions that are necessary to get the analytical solution is presented. In general, the analytical solutions appear in multiplets corresponding to the quantum number n of the harmonic oscillator. As an application, known solutions for the anharmonic oscillators are critically recalculated and a few additional results are found. Analytical solutions are also found for the generalized Morse oscillators.

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I. INTRODUCTION

The solution of the one-dimensional Schrödinger equation represents an important problem with numerous applications in many fields of physics. This equation can always be solved numerically. Despite this, analytical solutions yield a more detailed and exact description of the physical reality and are therefore of considerable interest.

The number of potentials $V(x)$ for which the analytic solution of the one-dimensional Schrödinger equation,

$$H\psi(x)=E\psi(x), \tag{1}$$

with the Hamiltonian

$$H=-\frac{d^2}{dx^2}+V(x) \tag{2}$$

is known is rather limited. Except for trivial cases, examples of analytically solvable problems include the harmonic oscillator, some anharmonic oscillators [1–9], the one-dimensional hydrogen atom, the Morse oscillator [10], and some other simple cases (see, e.g, [11–14]).

Analyzing these analytic solutions, we conclude that the bound-state wave functions have the same structure. The wave functions have the form of the exponential or other related functions multiplied by a polynomial in a variable that is a function of x. In other words, the wave functions ψ for all these problems can be written as a linear combination of functions $\psi_m=f^m g$, where $f(x)$ and $g(x)$ are suitably chosen functions and m is an integer.

It is obvious that there is a chance of finding an analytical solution if the Hamiltonian transforms the set of the basis functions ψ_m into itself. Namely, if the result of $H\psi_m$ is a finite linear combination of ψ_m, we can hope that the resulting finite order matrix problem is analytically solvable. Assuming these properties of the wave function and Hamiltonian, we discuss in this paper conditions for the functions f and g and the potential V, which must be fulfilled to get the analytical solution of the Schrödinger equation.

Using the approach indicated above we first use the basis ψ_m to transform the Schrödinger equation to the matrix form with a non-Hermitian matrix (Sec. II). Possible forms of f, g, and V that can yield analytical solutions are discussed in Sec. III. In the next three sections, known analytical results for the anharmonic oscillators are critically recalculated. Section IV is devoted to the problem of the quartic anharmonic oscillator. In Sec. V, a detailed analysis of the sextic oscillator is performed and a few new analytical solutions are found. Discussion of the higher-order anharmonic oscillators is presented in Sec. VI. Another interesting problem is the generalization of the Morse oscillator. The quadratic, quartic, sextic, and higher-order generalized Morse oscillators are investigated in Secs. VII–IX.

II. TRANSFORMATION OF THE SCHRÖDINGER EQUATION INTO THE MATRIX FORM

We assume the wave function ψ in the form

$$\psi=\sum_m c_m\psi_m, \tag{3}$$

where

$$\psi_m=(f)^m g. \tag{4}$$

The standard approach to the solution of the Schrödinger equation consists in substituting the assumption (3) into Eq. (1). Introducing the matrix elements

$$H_{mn}=\int \psi_m^* H\psi_n dx \tag{5}$$

and

$$S_{mn}=\int \psi_m^*\psi_n dx \tag{6}$$

one gets the well-known eigenvalue problem

$$\sum_n H_{mn}c_n = E\sum_n S_{mn}c_n. \tag{7}$$

In most cases, it appears impossible to calculate H_{mn} and S_{mn} and solve Eq. (7) analytically. However, if the matrices H and S are truncated, this method is suitable for calculating approximate solutions.

There is a chance of finding analytical solutions of the Schrödinger equation if the Hamiltonian H transforms the set of the basis functions ψ_m into itself. We assume therefore that the Hamiltonian H fulfills the relation

$$H\psi_m = \sum_n h_{mn}\psi_n, \tag{8}$$

where the coefficients h_{mn} are numbers. Let us introduce an overlap between the basis function ψ_m and the exact wave function ψ

$$M_m = \int \psi_m^* \psi dx. \tag{9}$$

Substituting Eqs. (8) and (9) into the Schrödinger equation (1) we get another matrix formulation known from the moment method [15–19]

$$\sum_n h_{mn}^* M_n = EM_m. \tag{10}$$

We see that the vector of the overlaps M_m is the right eigenvector of the matrix h. The advantage of Eq. (10) is that, in contrast to Eq. (7), there is no matrix S in this equation. The matrix h is usually sparse, which further simplifies the problem. On the other hand, the matrix h is non-Hermitian. The equations

$$H_{mn} = \sum_p S_{mp}h_{np} \tag{11}$$

and

$$M_m = \sum_n S_{mn}c_n \tag{12}$$

following from the assumptions (8) and (3) give the relation of the quantities appearing in Eqs. (7) and (10). Application of these equations is usually complicated by the infinite order of the matrix S.

There is also another possibility close to the approaches given above. If Eqs. (3) and (8) are used directly in the Schrödinger equation (1) the following result is obtained:

$$\sum_{m,n} c_m h_{mn}\psi_n = E\sum_m c_m\psi_m. \tag{13}$$

Assuming linear independence of the functions ψ_m we get a simple matrix problem,

$$\sum_m c_m h_{mn} = Ec_n. \tag{14}$$

The vector of the coefficients c_m is the left eigenvector of the matrix h.

The coefficients c_m are obtained from Eq. (14) directly without the necessity of using the transformation (12) as they are in the moment method.

Another disadvantage of the moment method is that even for the analytically solvable problems the overlaps M_m are usually different from zero and sometimes even diverge for $m\to\infty$ [9,15,16]. The problem (10) of the infinite order is difficult to solve analytically and even when it is solved, the transformation (12) of the usually infinite order must be applied. On the other hand, the left eigenvectors of Eq. (14) with a finite number of nonzero coefficients c_m can often be found directly and the analytical wave function can be found in the form of a finite linear combination of ψ_m. For the sake of simplicity, we discuss in this paper one-dimensional problems only. We note, however, that the moment method has successively been applied to one-dimensional as well as multidimensional problems (see, e.g. [16,17],).

The condition that only a finite number of the coefficients c_m is different from zero is known, for example, from the solution of the harmonic oscillator where c_m are the coefficients of the Hermite polynomials. In the standard solution of the harmonic oscillator a simple recurrence relation for the coefficients c_m of the Hermite polynomials is obtained. In our approach, such a simple recurrence relation is replaced by a general matrix equation (14) and can therefore lead to analytical solutions that have not been known until now.

A problem similar to Eq. (14) is solved also in the Hill determinant method (see, e.g. [5,20,21,13,14],). As we show below, our approach is more general than this method. We consider general functions f and g and give a general discussion of Eq. (14). We are also interested in a direct analytical solution of Eq. (14) for a finite linear combination in (3) instead of discussing the infinite-order problem.

The wave functions given in this paper are not normalized.

III. CONDITIONS FOR f AND g

In the previous section, the validity of Eq. (8) was assumed. Now we derive conditions for f,g, and V following from this assumption.

Applying the Hamiltonian (2) to the basis function (4) we get

$$H\psi_m = \left[-m(m-1)\frac{f'^2}{f^2} - m\left(2\frac{f'}{f}\frac{g'}{g} + \frac{f''}{f}\right) - \frac{g''}{g} + V\right]\psi_m. \tag{15}$$

Here, f' denotes df/dx.

In order to get $H\psi_m$ as a linear combination of ψ_n the expression in brackets must be a linear combination of f^n. As different terms in Eq. (15) depend on m in a different way any of the terms $f'^2/f^2, 2(f'/f)(g'/g)+f''/f$, and $-g''/g+V$ must be a linear combination of f^n. It follows from the first and second terms that f' must be a linear combination of f^m,

$$f' = \sum_m f_m(f)^m, \tag{16}$$

where f_m are numbers. Analogously, the second and third terms lead to

$$g' = -g \sum_m g_m (f)^m, \tag{17}$$

where the minus sign on the right-hand side is chosen for further convenience. Finally, the last term gives

$$V = \sum_m V_m (f)^m. \tag{18}$$

We see that the potentials V considered in this paper must have the form given by Eq. (18). At the same time, the function $f(x)$ appearing in this equation must satisfy Eq. (16). These two conditions restrict possible forms of the potentials for which our method is applicable.

We note that there are a number of simple functions fulfilling Eq. (16) such as x, $\exp(x)$, $\coth(x)$, and $\cot(x)$. However, there are also more complex functions such as the orthogonal polynomials that can be used as the function f.

The coefficients f_m, g_m, and V_m are arbitrary until now. If the coefficients f_m and g_m are known, the functions f and g can be obtained by inverting

$$x(f) = \int \frac{1}{\Sigma_m f_m (f)^m} df \tag{19}$$

and calculating

$$\begin{aligned} g(x) &= \exp\left(-\int \sum_m g_m (f)^m dx \right) \\ &= \exp\left(-\int \frac{\Sigma_m g_m (f)^m}{\Sigma_m f_m (f)^m} df \right). \end{aligned} \tag{20}$$

To get Eq. (14), the function g cannot be arbitrary and is given by Eq. (20), where g_m are parameters. The way to determine the coefficients g_m is described below.

In the moment method and the Hill determinant method the function (20) is often replaced by a single Gaussian exponential. Obviously, such an approximate approach cannot be used if analytic solutions are to be found.

As a result of the integration, the function $g(x)$ can have a rather complex form. It shows that the assumption about the polynomial form of the argument of the exponential made in the Hill determinant method is too restrictive (see the sections devoted to the generalized Morse potentials).

There is also another conclusion following from Eq. (20). Let us assume that we search for the bound-state wave function in the form of a finite sum (3). Then, investigating the integral in Eq. (20), it is easy to determine g_m for which $g(x)$ is finite. For example, let us assume that $f(x) = x$, $g_m \neq 0$ for $m = M$ and $g_m = 0$ for $m < 0$ and $m > M$. It follows from Eq. (20) that M must be odd, otherwise the function $g(x)$ diverges for $x \to \infty$ or $x \to -\infty$. In fact, this is the reason for which the analytical solutions for the quartic anharmonic oscillator with $M = 2$ cannot have this form of $g(x)$ (see Sec. IV).

Substituting Eqs. (16)–(18) into Eq. (15) we get

$$\begin{aligned} H\psi_m = \sum_i \Bigg[& -m(m-1) \sum_j f_j f_{i-j+2} - m \sum_j (-2 f_j g_{i-j+1} \\ & + j f_j f_{i-j+2}) - \sum_j (g_j g_{i-j} - j f_{i-j+1} g_j) + V_i \Bigg] \psi_{m+i}. \end{aligned}$$

Therefore, the matrix h_{mn} appearing in Eq. (14) equals

$$\begin{aligned} h_{m,m+i} = & -m(m-1) \sum_j f_j f_{i-j+2} - m \sum_j (-2 f_j g_{i-j+1} \\ & + j f_j f_{i-j+2}) - \sum_j (g_j g_{i-j} - j f_{i-j+1} g_j) + V_i. \end{aligned} \tag{21}$$

Our method of finding analytical solutions of the Schrödinger equation can be described as follows. First we determine the function $f(x)$ from the form of the potential $V(x)$ [see Eq. (18)]. Then we try to find the coefficients g_m and V_m for which the left eigenvectors of the matrix h exist with a finite number of nonzero components. This leads to a solution of a system of equations for g_m and V_m, which is often possible to solve. If the analytical solutions of Eq. (14) are found the wave functions are determined from Eqs. (3) and (20).

We note that the boundary conditions for the wave function have not been taken into consideration until now. This means that this method can be used for the discrete as well as continuous part of the energy spectrum. It also means that to get wave functions for the discrete energies, only the solutions satisfying the appropriate boundary conditions must be taken.

In general, solution of Eq. (14) leads to two linearly independent solutions as it should be for the differential equation of the second order. For the bound states, only one of the solutions or their suitable linear combination must be taken.

Now we search for the left eigenvector of the matrix h with a finite number of nonzero components. In this paper, we assume $c_m = 0$ for $m < 0$ and $m > n$, where $n \geqslant 0$ is an integer. It means that we search for the wave function in the form

$$\psi = \sum_{m=0}^{n} c_m f^m g. \tag{22}$$

If necessary, the summation in this equation can be extended to $m < 0$.

The corresponding eigenvalue problem (14) becomes

$$\sum_{m=0}^{n} c_m (h_{m,m+i} - E\,\delta_{m,m+i}) = 0, \tag{23}$$

where $i = \ldots, -2, -1, 0, 1, 2, \ldots$. This formula represents more equations than the number of unknown coefficients c_m and has in general only the trivial solution $c_m = 0$, $m = 0, \ldots, n$. To get nonzero c_m, the number of equations must be reduced or they must be made linearly dependent.

Our aim is to reduce the problem (23) to a standard eigenvalue problem with a square matrix.

General discussion of this problem is rather complex. In this paper, we assume the potential in the form

$$V=\sum_{i=1}^{2M} V_i(f)^i. \qquad (24)$$

If necessary, negative powers $i<0$ can be also included. The potential coefficients V_1, ..., V_{2M} appear in $h_{m,m+i}$, $i=1,\ldots,2M$. Assuming further $g_m=f_m=0$ for $m<0$ and $m>M$, the matrix $\{h_{ij}\}$ has nonzero elements in the rows $i=0,\ldots,\ n$ and columns $j=0,\ldots,\ n+2M$. To reduce the number of columns, we start with the last one $j=n+2M$ and determine g_M in such a way that the only nonzero element $h_{n,n+2M}$ in this column becomes zero. This leads to $g_M^2=V_{2M}$ so that $g_M=\pm\sqrt{V_{2M}}$.

Let us assume for a moment that the potential is quadratic ($M=1$). In this case we calculate g_0 from the condition that the remaining nonzero element $h_{n,n+1}$ in the $(n+1)$th row equals zero. As a result, the eigenvalue problem (23) with a square matrix is obtained. We see that the problem of the quadratic oscillators can be solved easily.

For quartic and higher-order potentials ($M=2,3,\ldots$), however, we get more nonzero elements in the columns $j=n+2M-1,\ldots,n+1$ than in the case of the quadratic oscillators. In this case, g_{M-1}, ..., g_0 must be determined from the condition that the columns $j=n+2M-1,n+2M-2,\ldots,\ n+M$ are linearly dependent on the columns $j=0,\ldots,\ n$ of the matrix $h-E$. To reduce the number of linearly independent columns of h, we must continue to introduce some constraints on the potential coefficients that were arbitrary until now. Considering the columns $j=n+M-1,\ldots,n+1$ we can calculate V_{M-1}, ..., V_1 as a function of V_M, ..., V_{2M}. Solving then the remaining problem (23) with the square matrix $\{h_{ij}\}$, $i,j=0,\ldots,\ n$ we can find the solution in the form (22). We see that the analytic solution in the form (22) exists for nonquadratic potentials only if additional constraints on the potential coefficients are introduced.

We note that, in general, the values of g_0, ..., g_M and V_1, ..., V_{M-1} depend on the energy E and n. For $n=0$, we can find only one analytical solution with the corresponding values of g_0, ..., g_M and V_1, ..., V_{M-1}. Then we can get analytical solutions for $n=1$, etc. Thus, the solutions are obtained in certain multiplets corresponding to different values of n. Our n corresponds to the quantum number n of the harmonic oscillator for which the matrix h can be easily diagonalized and the energies $E_n=(2n+1)g_1-g_0^2=\hbar\omega(n+1/2)$ are obtained.

In general, the best chance to find the analytical solution is for $n=0$ when the matrix h reduces to one row. The coefficients g_m are then given by equations $h_{0j}=0$, $j=2M$, ..., M and the potential constraints follow from $h_{0j}=0$, $j=M-1,\ldots,1$. The energy equals $E=h_{00}$ and the corresponding wave function is $\psi(x)=g(x)$. With increasing n and M, the order of the problem and complexity of the potential constraints increase and the chance to find explicit analytic expressions for the energies and wave functions is lower. In general case, a numerical solution of the problem (23) is necessary.

Let us discuss now the case of the anharmonic and Morse oscillators. For the anharmonic oscillators we put $f(x)=x$, $f_m=\delta_{m,0}$ and for the generalized Morse oscillators we use $f(x)=1-\exp(-x)$, $f_0=1,f_1=-1$ and $f_m=0$ otherwise. The potential is assumed in the form (24). As follows from our discussion given above, analytical solutions for the anharmonic oscillators exist only if M is odd, i.e., if $2M=4k+2$, where k is an integer. On the other hand, analytical solutions for the generalized Morse oscillators exist for any M. The way to solve the problem (23) is the same for both types of oscillators. First, we choose n from the range $n=0,1,\ldots$. Then we solve the equation $h_{n,n+2M}=0$ leading to $g_M^2=V_{2M}$. After that we continue with the solution of the equations $h_{n,n+i}=0$, $i=2M-1,\ldots,M$, which yield g_{M-1}, ..., g_0 as a function of V_M, ..., V_{2M}. Consequently, all the coefficients g_m are determined and all columns of the matrix h, $j=n+2M$, ..., $n+M$ are equal to zero. Then we continue with the columns $j=n+M-1,\ldots,n+1$ and determine the corresponding constraints on the potential coefficients $V_{M-1},\ldots,V_1$. The total number of the nonzero coefficients g_m $(M+1)$ plus the number of the potential constraints $(M-1)$ equals $2M$. If the potential is even, the number of the constraints reduces to one-half.

A less general discussion was performed in [8] for the anharmonic oscillators with the even potential.

The discussion given above shows that all the analytically solvable problems with the wave function in the form of a finite linear combination (3) have the same algebraic structure given by the matrix (21). If the function f is changed the general discussion regarding h, g, g_m, and V_m remains the same. Assuming that the potential coefficients V_m, $m=M$, ..., $2M$ remain unchanged for new f we get new values of g_m and potential constraints on V_m, $m=1,\ldots,M-1$. However, because of the integration in Eq. (20), the function g and the wave function ψ can change considerably.

IV. QUARTIC ANHARMONIC OSCILLATOR

The potential has the form

$$V(x)=V_1x+V_2x^2+V_3x^3+V_4x^4, \quad V_4>0$$

corresponding to $M=2$. Assuming $g_m\neq 0$ for $m=0,1,2$ and $f(x)=x$ the matrix h equals

$$h_{m,m+i}=-m(m-1)\,\delta_{i,-2}+2mg_0\delta_{i,-1}+(2mg_1-g_0^2+g_1)\,\delta_{i,0}$$
$$+(2mg_2-2g_1g_0+2g_2+V_1)\,\delta_{i,1}+(-2g_2g_0-g_1^2+V_2)\,\delta_{i,2}+(-2g_1g_2+V_3)\,\delta_{i,3}+(-g_2^2+V_4)\,\delta_{i,4}.$$

First we discuss the ground state corresponding to $n=0$. The most simple wave function with no nodes is given by the left eigenvector $c_m=\delta_{m,0}$ so that $\psi(x)=g(x)$. To find ψ it is sufficient to find g_m and the potential constraint on V_1 for which $h_{0,i}=0, i=4,\ldots,\ 1$. Two possible solutions of these equations are as follows. The coefficients g_m are given by

$$g_2=\pm\sqrt{V_4},\quad g_1=V_3/(2g_2),\quad g_0=(V_2-g_1^2)/(2g_2)$$

and the potential constraint giving V_1 as a function of $V_2,\ \ldots,\ \ V_4$ is

$$V_1=2g_1g_0-2g_2.$$

The energy E equals

$$E=h_{00}=g_1-g_0^2.$$

It can easily be verified that both functions

$$\psi(x)=g(x)=\exp(-g_0x-g_1x^2/2-g_2x^3/3) \tag{25}$$

for $g_2=\pm\sqrt{V_4}$ satisfy the Schrödinger equation (1). However, they diverge for $x\to\infty$ or $x\to-\infty$, as concluded in the previous section.

For the higher multiplets $n>0$ the situation is analogous. We see therefore that the wave functions of the quartic anharmonic oscillator cannot have the form (22) with $g(x)$ given by Eq. (25).

V. SEXTIC ANHARMONIC OSCILLATOR

The potential is assumed in the form

$$V(x)=V_1x+\cdots+V_6x^6,\quad V_6>0.$$

Assuming further $g_m\neq 0$, $m=0,\ldots,\ \ 3$ the matrix h becomes

$$\begin{aligned}h_{m,m+i}=&-m(m-1)\,\delta_{i,-2}+2mg_0\,\delta_{i,-1}+(2mg_1-g_0^2+g_1)\,\delta_{i,0}\\&+(2mg_2-2g_1g_0+2g_2+V_1)\,\delta_{i,1}+(2mg_3-2g_2g_0-g_1^2+3g_3+V_2)\,\delta_{i,2}\\&+(-2g_3g_0-2g_1g_2+V_3)\,\delta_{i,3}+(-2g_1g_3-g_2^2+V_4)\,\delta_{i,4}+(-2g_2g_3+V_5)\,\delta_{i,5}+(-g_3^2+V_6)\,\delta_{i,6}.\end{aligned}$$

A. $n=0$

The values of g_m and the potential constraints are found by solving successively $h_{0j}=0$, $j=6,\ldots,\ \ 1$.

The coefficients g_m equal

$$g_3=\sqrt{V_6},\quad g_2=V_5/(2g_3),\quad g_1=(V_4-g_2^2)/(2g_3),$$

$$g_0=(V_3-2g_1g_2)/(2g_3), \tag{26}$$

where the $+$ sign before $\sqrt{V_6}$ follows from the boundary conditions at $x\to\pm\infty$. These equations for g_m are also valid for all the higher multiplets $n=1,2,\ \ldots.$

The coefficients $V_3,\ \ldots,\ \ V_6$ can be arbitrary. Two remaining coefficients are given by the potential constraints

$$V_1=2g_1g_0-2g_2,\quad V_2=g_1^2+2g_2g_0-3g_3.$$

The corresponding energy and wave function equal

$$E=h_{00}=g_1-g_0^2$$

and

$$\psi(x)=\exp(-g_0x-g_1x^2/2-g_2x^3/3-g_3x^4/4).$$

This function has no nodes and is therefore the ground-state wave function. We see that the analytic solution exists for the asymmetric potential with general potential coefficients $V_3,\ \ldots,\ \ V_6$. In [8], the solutions were found for the even potential only.

In a special case of the even potential,

$$V(x)=V_2x^2+V_4x^4+V_6x^6,$$

much more simple formulas are obtained,

$$V_2=V_4^2/(4V_6)-3\sqrt{V_6},$$

$$E=V_4/(2\sqrt{V_6}),$$

and

$$\psi(x)=\exp[-V_4x^2/(4\sqrt{V_6})-\sqrt{V_6}x^4/4].$$

This result has one more parameter than the example given in [1]. These equations give the ground state of the sextic double-well potential. If $V_4<0$, the energy E lies below the maximum of the potential at $x=0$ and the wave function has two maxima at $x=\pm\sqrt{-V_4/(2V_6)}$.

B. $n=1$

In this case, we solve successively the equations

$$\sum_{m=0}^{1}c_m(h_{mj}-E\,\delta_{mj})=0 \tag{27}$$

for $j=7,\ldots,\ \ 0$. First we solve these equations for $j=7,\ldots,\ \ 4$. This leads to Eqs. (26). Then, Eq. (27) for $j=3$ gives

$$V_2=g_1^2+2g_2g_0-5g_3.$$

Assuming for simplicity $c_1=1$ we get from Eq. (27) for $j=2$

$$c_0=-h_{12}/h_{02}=(V_1-2g_1g_0+4g_2)/(2g_3).$$

Then we solve two equations following from Eq. (27) for $j=0$ and $j=1$ and get the cubic equation for V_1

$$V_1^3+(10g_2-6g_1g_0)V_1^2+(32g_2^2+4g_1g_3+12g_1^2g_0^2 -40g_1g_0g_2)V_1+32g_2^3-8g_3^2g_0-8g_1^3g_0^3-64g_1g_0g_2^2 -8g_1^2g_0g_3+16g_1g_2g_3+40g_1^2g_0^2g_2=0.$$

Thus, depending on the values of V_3, ..., V_6, we can get up to three real values of V_1 for which the analytical solution of the Schrödinger equation exists. The corresponding energy obtained from Eq. (27) for $j=0,1$ equals

$$E=(V_1^2+(6g_2-4g_1g_0)V_1+4g_1^2g_0^2-12g_1g_0g_2+8g_2^2 +6g_1g_3-2g_0^2g_3)/(2g_3)$$

and the wave function is

$$\psi(x)=(c_0+c_1x)\exp(-g_0x-g_1x^2/2-g_2x^3/3-g_3x^4/4).$$

This function has one node and represents the first excited-state wave function.

In a special case $c_0=0$ a more simple result with three potential constraints instead of two is obtained. The potential constraints are

$$V_1=-4g_2,\quad V_2=-5g_3+g_1^2,\quad V_3=2g_1g_2.$$

The last constraint leads to $g_0=0$. The energy and wave function with one node corresponding to this potential equal

$$E=3g_1$$

and

$$\psi(x)=x\exp(-g_1x^2/2-g_2x^3/3-g_3x^4/4). \tag{28}$$

In [8], a special analytic solution corresponding to Eq. (28) for the even potential was given. In this paper, we have found solutions for a more general asymmetric potential.

C. $n=2$

General discussion leads to rather complicated expressions that will not be given here. We discuss only the special case $c_0\neq 0, c_1=0, c_2\neq 0$. Analyzing the equations

$$\sum_{m=0}^{2} c_m(h_{mj}-E\delta_{mj})=0,\quad j=0,...,\quad 8 \tag{29}$$

we get conditions $g_0=g_2=0$. It follows from these equations that the potential $V(x)$ must be even,

$$V=V_2x^2+V_4x^4+V_6x^6.$$

The same form of the potential also will be assumed for the higher-order multiplets. Because of the symmetry of the potential the number of potential constraints reduces to one,

$$V_2=g_1^2-7g_3.$$

There are two energies,

$$E_\pm=3g_1\pm 2\sqrt{g_1^2+2g_3}$$

and wave functions

$$\psi_\pm(x)=[1+(g_1-E_\pm)x^2/2]\exp(-g_1x^2/2-g_3x^4/4)$$

solving the Schrödinger equation in this case. The $-$ sign denotes the ground state (the wave function ψ_- has no nodes). The $+$ sign denotes the second excited state (the wave function has two nodes). We also see that $E_-<E_+$.

D. $n=3$

Now we search for a special solution of Eq. (14) corresponding to $c_m\neq 0$ for $m=1,3$ (the solution with the odd parity) and the even potential. We get the following result:

$$V_2=g_1^2-9g_3,$$

$$E_\pm=5g_1\pm 2\sqrt{g_1^2+6g_3}$$

and

$$\psi_\pm(x)=x[1+(3g_1-E_\pm)x^2/6]\exp(-g_1x^2/2-g_3x^4/4).$$

The $-$ sign denotes the first excited state (the corresponding wave function has one node). The $+$ sign denotes the third excited state (the wave function has three nodes).

E. Higher-order multiplets for n even

The energies and wave functions corresponding to the even potential are given by the constraint

$$V_2=g_1^2-(2n+3)g_3$$

and eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} g_1 & -2ng_3 & 0 & 0 & 0 & 0\\ -2 & 5g_1 & (-2n+4)g_3 & 0 & 0 & 0\\ 0 & -12 & 9g_1 & (-2n+8)g_3 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 0 & -(n-2)(n-3) & (2n-3)g_1 & -4g_3\\ 0 & \cdots & 0 & 0 & -n(n-1) & 2n(+1)g_1 \end{pmatrix}.$$

The left eigenvectors of this matrix with the components c_0, c_2, ..., c_n give $n/2+1$ even wave functions

$$\psi(x)=\sum_{m=0}^{n/2} c_{2m}x^{2m}\exp(-g_1x^2/2-g_3x^4/4).$$

F. Higher-order multiplets for n odd

Again, we assume the even potential. The energies and wave functions corresponding to the potential constraint

$$V_2=g_1^2-(2n+3)g_3$$

are given by the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 3g_1 & (-2n+2)g_3 & 0 & 0 & 0 & 0 \\ -6 & 7g_1 & (-2n+6)g_3 & 0 & 0 & 0 \\ 0 & -20 & 11g_1 & (-2n+10)g_3 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -(n-2)(n-3) & (2n-3)g_1 & -4g_3 \\ 0 & \cdots & 0 & 0 & -n(n-1) & 2n(+1)g_1 \end{pmatrix}.$$

The left eigenvectors of this matrix with the components $c_1,c_3,\ldots,\ c_n$ give $(n+1)/2$ odd wave functions:

$$\psi(x)=\sum_{m=0}^{(n-1)/2} c_{2m+1}x^{2m+1}\exp(-g_1x^2/2-g_3x^4/4).$$

VI. HIGHER-ORDER ANHARMONIC OSCILLATORS

As shown in Sec. III, analytically solvable anharmonic oscillators are only those with the highest-order term x^{4k+2}, where k is an integer.

The solution of the problem of the higher-order oscillators is analogous to that for the sextic oscillator. As an example we consider the decadic oscillator with

$$V=V_1x+\cdots+V_{10}x^{10},\quad V_{10}>0.$$

Assuming $g_m\neq 0$ for $m=0,\ldots,\ 5$ the matrix h has the form

$$\begin{aligned} h_{m,m+i}=&-m(m-1)\delta_{i,-2}+2mg_0\delta_{i,-1}+(2mg_1-g_0^2+g_1)\delta_{i,0}\\ &+(2mg_2-2g_1g_0+2g_2+V_1)\delta_{i,1}+(2mg_3-2g_2g_0-g_1^2+3g_3+V_2)\delta_{i,2}\\ &+(2mg_4-2g_3g_0-2g_1g_2+4g_4+V_3)\delta_{i,3}+(2mg_5-2g_1g_3-2g_4g_0-g_2^2+5g_5+V_4)\delta_{i,4}\\ &+(-2g_1g_4-2g_2g_3-2g_5g_0+V_5)\delta_{i,5}+(-2g_1g_5-2g_2g_4-g_3^2+V_6)\delta_{i,6}\\ &+(-2g_3g_4-2g_2g_5+V_7)\delta_{i,7}+(-2g_3g_5-g_4^2+V_8)\delta_{i,8}+(-2g_4g_5+V_9)\delta_{i,9}+(-g_5^2+V_{10})\delta_{i,10}. \end{aligned}$$

Solving equations $h_{n,n+i}=0$ for $i=10,\ldots,5$ the following values of g_m are obtained:

$$g_5=\pm\sqrt{V_{10}},\quad g_4=V_9/(2g_5),\quad g_3=(V_8-g_4^2)/(2g_5),$$

$$g_2=(V_7-2g_3g_4)/(2g_5),\quad g_1=(V_6-g_3^2-2g_2g_4)/(2g_5)$$

and

$$g_0=(V_5-2g_2g_3-2g_1g_4)/(2g_5).$$

Similarly to the sextic anharmonic oscillator we take $g_5=\sqrt{V_{10}}$. These values are the same for all the multiplets. Let us consider for example $n=0$. Then, the potential constraints following from $h_{0,i}=0, i=4,\ldots,\ 1$ are

$$V_1=2g_1g_0-2g_2,\quad V_2=2g_2g_0-3g_3+g_1^2,$$

$$V_3=2g_3g_0+2g_1g_2-4g_4,$$

$$V_4=2g_1g_3+2g_4g_0-5g_5+g_2^2.$$

Again, the analytic solution exists for the asymmetric potential. The ground-state wave function of the singlet $n=0$ is given by

$$\psi(x)=\exp\left(-\sum_{m=1}^{6} g_{m-1}x^m/m\right).$$

The corresponding energy equals

$$E=g_1-g_0^2.$$

Results for the higher-order multiplets are analogous to those for the sextic oscillator and will not be given here.

VII. QUADRATIC MORSE OSCILLATOR

The Morse oscillator [10] with the potential

$$V(r)=D\{1-\exp[-(r-r_0)/a]\}^2 \tag{30}$$

is of considerable interest in molecular physics. In this paper, we use the variable $x=(r-r_0)/a$ and discuss generalized Morse potentials in the form

$$V(x)=\sum_{i=1}^{2M} V_i[1-\exp(-x)]^i. \tag{31}$$

Such potentials are more general than the original Morse potential and can describe, for example, potentials with resonances when the barrier higher than the value of the potential at $x\rightarrow\infty$ exists. As we pointed out in Sec. III, in case of the generalized Morse oscillators we are not limited by the $2M=4k+2$ rule valid for the anharmonic oscillators and M can be an arbitrary positive integer.

We take now $f(x)=1-\exp(-x)$ so that $f_0=1,f_1=-1$, and $f_m=0$ otherwise.

First we discuss briefly the quadratic Morse oscillator with the potential

$$V=V_1[1-\exp(-x)]+V_2[1-\exp(-x)]^2,\quad V_2>0, \tag{32}$$

which is equivalent to the original Morse potential (30).

For the quadratic Morse oscillator ($M=1$) there are no potential constraints so that all the multiplets $n=0,1,\ldots$ belong to the same potential.

Assuming $g_m=0$ for $m>1$ the matrix h becomes

$$\begin{aligned} h_{m,m+i}=&-m(m-1)\,\delta_{i,-2}+m(2m-1+2g_0)\,\delta_{i,-1}\\ &+[-m(m+2g_0-2g_1)+g_1-g_0^2]\,\delta_{i,0}\\ &+[-g_1(2m+2g_0+1)+V_1]\,\delta_{i,1}+(V_2-g_1^2)\,\delta_{i,2}. \end{aligned}$$

Taking into account the expression for $g(x)$,

$$\begin{aligned} g(x)=\exp\left(-\int[g_0+g_1f(x)]dx\right)=\exp[-(g_0+g_1)x\\ -g_1\exp(-x)] \end{aligned}$$

the value

$$g_1=\sqrt{V_2}$$

must be taken. Similarly, to get $h_{n,n+1}=0$ for a given n, the value

$$g_0=V_1/(2g_1)-1/2-n$$

must be used. In contrast to the anharmonic oscillators, g_0 is a function of n. In order to get bound states the wave function ψ must be finite for $x\rightarrow\pm\infty$. It follows from

$$\psi(x)=\sum_{m=0}^{n} c_m[1-\exp(-x)]^m g(x)$$

that to fulfill these boundary conditions the relation

$$g_0+g_1>0$$

must be valid. Taking into account the form of $g_0=g_0(n)$ we see that there is a maximum value of $n=n_{\max}$ for which the boundary conditions are obeyed. We get

$$n_{\max}=[V_1/(2g_1)-1/2+g_1],$$

where [] denotes the integer part. Therefore, only a finite number of bound states for $n=0,\ldots,\quad n_{\max}$ exists. There are no bound states for $V_1/(2g_1)-1/2+g_1\leqslant 0$.

To get the eigenvalues we assume $h_{n,n+1}=h_{n,n+2}=0$ for a given n and make use of the summation rule

$$\sum_i h_{m,m+i}=(2n+1)g_1-g_0^2.$$

This equation shows that the energies

$$E_n=(2n+1)g_1-g_0^2=(2n+1)g_1-[V_1/(2g_1)-(n+1/2)]^2 \tag{33}$$

are the eigenvalues of the matrix $\{h_{ij}\}$, $i,j=0,\ldots,\quad n$ since the columns of the matrix $\{h_{ij}\}-E$, $i,j=0,\ldots,\quad n$ are linearly dependent. It is worth noting that, except for the expression for g_0, Eq. (33) is the same as that for the energy of the harmonic oscillator with the potential $V=V_1x+V_2x^2$.

To get corresponding c_m we solve Eq. (23), leading to the following system of recurrence equations:

$$\begin{gathered} c_n=1,\\ c_{n-1}h_{n-1,n}+c_n(h_{n,n}-E)=0,\\ c_{n-2}h_{n-2,n-1}+c_{n-1}(h_{n-1,n-1}-E)+c_nh_{n,n-1}=0,\\ c_{i-1}h_{i-1,i}+c_i(h_{i,i}-E)+c_{i+1}h_{i+1,i}+c_{i+2}h_{i+2,i}=0,\\ i=n-2,\ldots,1,\\ c_0(h_{0,0}-E)+c_1h_{10}+c_2h_{20}=0. \end{gathered} \tag{34}$$

It can be shown that the results of this section agree with known results for the standard Morse oscillator with the potential (30).

VIII. QUARTIC MORSE OSCILLATOR

Now we discuss the quartic oscillator with the potential (31) for $M=2$. For the quartic and higher-order Morse oscillators, we write the function g as

$$g(x)=\exp\left(-\sum_{m=0}^{M} g_mG_m(x)\right),$$

where

$$G_m(x)=\int [1-f(x)]^m dx.$$

These functions equal for the quartic oscillator

$$G_0(x)=x,\quad G_1(x)=x+\exp(-x),$$

$$G_2(x)=x+2\exp(-x)-\exp(-2x)/2.$$

The matrix h for the quartic oscillator is given by the formula

$$\begin{aligned}h_{m,m+i}=&-m(m-1)\delta_{i,-2}+(2m^2-m+2mg_0)\delta_{i,-1}+(-m^2-2mg_0+2mg_1-g_0^2+g_1)\delta_{i,0}\\&+(-2mg_1+2mg_2-2g_1g_0+2g_2-g_1+V_1)\delta_{i,1}+(-2mg_2-2g_2g_0-g_1^2-2g_2+V_2)\delta_{i,2}\\&+(-2g_1g_2+V_3)\delta_{i,3}+(-g_2^2+V_4)\delta_{i,4}.\end{aligned}$$

Solving successively $h_{n,n+i}=0, i=4,..., \quad 2$ for a given n we get the coefficients g_m:

$$g_2=\pm\sqrt{V_4},\quad g_1=V_3/(2g_2),$$

$$g_0=(V_2-g_1^2)/(2g_2)-n-1. \tag{35}$$

To get bound states, we take

$$g_2=-\sqrt{V_4},$$

where $V_4>0$. A further obvious condition for the existence of the bound states is

$$g_0+g_1+g_2>0.$$

The expression for the maximum n giving the bound states is as follows:

$$n_{\max}=[(V_2-g_1^2)/(2g_2)-1+g_1+g_2].$$

If the argument of the integer part is less than or equal to zero, there are no bound states.

Assuming that n is given, the summation rule for the quartic Morse oscillator equals

$$\sum_i h_{m,m+i}=2(n+1)g_2-g_0^2-2g_1g_0+V_1.$$

Similarly to the case of the quadratic Morse oscillator, this equation shows that the energies of the quartic Morse oscillator equal

$$E_n=2(n+1)g_2-g_0^2-2g_1g_0+V_1. \tag{36}$$

A. $n=0$

For $n=0$, we solve the condition $h_{01}=0$ and get the potential constraint

$$V_1=2g_1g_0-2g_2+g_1.$$

The energy and ground-state wave functions are given by

$$E=h_{00}=g_1-g_0^2 \tag{37}$$

and

$$\psi(x)=\exp\left(-\sum_{m=0}^{2} g_mG_m(x)\right). \tag{38}$$

B. $n=1$

For $n=1$ we proceed similarly as in the case of the sextic anharmonic oscillator. The system of equations (27) must be fulfilled for $j=5,..., \quad 0$. The equations for $j=5,4,3$ are valid because of (35). Assuming $c_1=1$, we calculate c_0 from Eq. (27) for $j=2$

$$c_0=-(V_1-3g_1+4g_2-2g_1g_0)/(2g_2).$$

Equation (27) for $j=1$ gives the potential constraint

$$V_1=-4g_2+2g_1+2g_1g_0\pm\sqrt{g_1^2-4g_2g_0-2g_2}. \tag{39}$$

Equation (27) for $j=0$ is satisfied since the energy (36) for $n=1$

$$E=4g_2-g_0^2-2g_1g_0+V_1$$

is the eigenvalue. The corresponding wave function equals

$$\psi(x)=\{c_0+c_1[1-\exp(-x)]\}\exp\left(-\sum_{m=0}^{2} g_mG_m(x)\right).$$

Because of Eq. (39), we can get two wave functions. One function has no nodes and the other has one node.

There is also a special solution corresponding to $c_m=\delta_{m1}$. This assumption leads to the additional potential constraint $2g_0+1=0$ or

$$V_2=V_3^2/(4V_4)-3\sqrt{V_4}.$$

The energy and wave function with one node equal in this case

$$E=3g_1-g_0^2$$

and

$$\psi(x)=[1-\exp(-x)]\exp\left(-\sum_{m=0}^{2} g_mG_m(x)\right).$$

C. $n=2$

Similarly to case $n=1$, we solve Eq. (29) for $j=6,\ldots,0$. The equations for $j=6,5,4$ are satisfied because of Eq. (35). Assuming $c_2=1$ we first solve Eq. (27) for $j=3$ and then for $j=2$. This leads to expressions for c_1 and c_0. Substituting these expressions to Eq. (23) for $j=1$, we get the cubic equation for V_1. The resulting expressions for V_1, c_1, and c_0 are complex and will not be given here. The energy is given by Eq. (36) and the wave function equals

$$\psi(x)=\{c_0+c_1[1-\exp(-x)]+c_2[1-\exp(-x)]^2\} \times\exp\left(-\sum_{m=0}^{2} g_m G_m(x)\right).$$

In this case, up to three analytical solutions can be obtained. These solutions have one, two, and three nodes.

Similarly to case $n=1$ a special solution with $c_0\neq 0$, $c_1=0$, and $c_2\neq 0$ corresponding to two additional potential constraints

$$g_1=g_2, \quad 2g_0+3=0$$

exists. Detailed discussion will not be given here.

D. Higher-order multiplets

The solution of Eq. (23) for higher n can be obtained in a similar way as described above. However, the results are complex and in general the numerical solution of Eq. (23) is necessary.

E. Transition to anharmonic oscillator

The transition from the quartic Morse potential to the quartic anharmonic potential can be made if the function

$$f(x)=[1-\exp(-ax)]/a\approx x-ax^2/2+a^2x^3/6-\cdots,$$

where $a\to 0+$ is used. The function g equals in this case

$$g(x)=\exp\{-g_0x-g_1[x/a+\exp(-ax)/a^2] -g_2[x/a^2+2\exp(-ax)/a^3-\exp(-2ax)/(2a^3)]\}.$$

To get finite $g(x)$ for $x\to-\infty$ we use

$$g_2=-\sqrt{V_4}.$$

From the same condition at $x\to\infty$ we get

$$g_0+g_1/a+g_2/a^2>0.$$

Using the coefficients

$$g_0=(V_2-g_1^2)/(2g_2)-a(n+1)$$

and

$$g_1=V_3/(2g_2)$$

the last condition becomes

$$[V_3^2/(4V_4)-V_2]/(2\sqrt{V_4})-a(n+1)-V_3/(2a\sqrt{V_4}) >\sqrt{V_4}/a^2.$$

For $a>0$, this condition can be fulfilled only for certain values of n, $n=0,\ldots,n_{\max}$. It is obvious that for $a\to 0+$ $n_{\max}$ is less than zero and in agreement with our conclusion in Sec. IV there are no bound states in the form assumed in this paper.

IX. SEXTIC AND HIGHER-ORDER MORSE OSCILLATOR

For the sextic oscillator $M=3$ and the function G_3 equals

$$G_3(x)=\int[1-f(x)]^3dx=x+3\exp(-x)-3\exp(-2x)/2 +\exp(-3x)/3.$$

The matrix h for the sextic oscillator is

$$\begin{aligned}h_{m,m+i}=&-m(m-1)\delta_{i,-2}+(2m^2-m+2mg_0)\delta_{i,-1}+(-m^2+2m(g_1-g_0)-g_0^2+g_1)\delta_{i,0}\\&+(-2mg_1+2mg_2-2g_1g_0+2g_2-g_1+V_1)\delta_{i,1}+(2m(g_3-g_2)-2g_2g_0-g_1^2-2g_2+3g_3+V_2)\delta_{i,2}\\&+(-2mg_3-2g_1g_2-2g_0g_3-3g_3+V_3)\delta_{i,3}+(-2g_1g_3-g_2^2+V_4)\delta_{i,4}\\&+(-2g_2g_3+V_5)\delta_{i,5}+(-g_3^2+V_6)\delta_{i,6},\end{aligned}$$

where $g_m=0$ for $m>3$ is assumed.

Solving successively $h_{n,n+i}=0, i=6,\ldots,3$ we get the coefficients g_m for a given n

$$g_3=\pm\sqrt{V_6}, \quad g_2=V_5/(2g_3), \quad g_1=(V_4-g_2^2)/(2g_3)$$

and

$$g_0=(V_3-2g_1g_2)/(2g_3)-n-3/2.$$

To get the bound states, we take

$$g_3=\sqrt{V_6},$$

where $V_6>0$. Further condition for the existence of the bound states is

$$g_0+g_1+g_2+g_3>0.$$

The expression for the maximum n giving the bound states equals

$$n_{\max}=[(V_3-2g_1g_2)/(2g_3)-3/2+g_1+g_2+g_3].$$

If the argument of the integer part is less than or equal to zero, there are no bound states.

The summation rule for the sextic Morse oscillator,

$$\sum_i h_{m,m+i} = -g_0^2 - g_1^2 - 2g_1g_0 - 2g_2g_0 - 2g_1g_2 - 2g_3g_0 + V_1 + V_2 + V_3,$$

leads to the energies

$$E_n = -g_0^2 - g_1^2 - 2g_1g_0 - 2g_2g_0 - 2g_1g_2 - 2g_3g_0 + V_1 + V_2 + V_3,$$

where $g_0 = g_0(n)$ and constraints on V_1, V_2, V_3 depend also on n.

For example, for $n=0$ we get

$$V_1 = 2g_0g_1 - 2g_2 + g_1, \quad V_2 = g_1^2 + 2g_2g_0 + 2g_2 - 3g_3,$$

$$V_3 = 2g_1g_2 + 2g_3g_0 + 3g_3,$$

$$E = g_1 - g_0^2$$

and

$$\psi(x) = \exp\left(-\sum_{m=0}^{3} g_m G_m(x)\right).$$

The other calculations for the sextic and higher-order Morse oscillators are analogous to that for the quadratic Morse oscillator. They will not be given here.

X. CONCLUSIONS

In this paper, a method for calculating the analytic solutions of the Schrödinger equation similar to the moment method and the Hill determinant method has been suggested.

First, the potential is assumed in the form $V(x) = \Sigma_m V_m f^m$, where $f = f(x)$ is a function that must satisfy certain conditions described below. In general, the summation can also run over the negative values of m. Then, the wave function is assumed to be a finite linear combination of the functions $\psi_m = f^m g$, where $g = g(x)$ is a convenient function. To get analytical solutions, it is assumed that the Hamiltonian transforms this basis set into itself. From the last assumption, we conclude that the derivative of f must be a finite linear combination of f^m with the coefficients f_m. The same condition must be valid for the logarithmic derivative of g, i.e., g'/g. For a given function f, the function g can easily be calculated from the equation $g(x) = \exp(-\int \Sigma_m g_m f^m dx)$, where g_m are constants. If the last expression and the expression for f' are used in the Schrödinger equation, a simple eigenvalue problem (14) with the matrix (21) is obtained. To get the analytic solution, the constants g_m must be determined in such a way that the analytic eigenvalues and left eigenvectors of this matrix exist. In general, some constraints on the potential coefficients also must be introduced. It appears that the solutions exist in multiplets corresponding to different values of the quantum number n of the harmonic oscillator. In general, different solutions correspond to different potentials.

Let us assume now that the potential has the form $V = \Sigma_m^{2M} V_m f^m$, $V_{2M} > 0$. It has been shown that the conditions for g_m necessary for the existence of bound states follow from the form of the function $g(x)$. For $f(x) = x$, analytic solutions exist only for $2M = 4k+2$, where k is an integer.

This method is a generalization of the approaches known from the moment method and the Hill determinant method and its main advantages are (1) known properties of $f(x)$ for which the analytical solution exist, (2) a formula for $g(x)$ with parameters g_m that can be found from the solution of the eigenvalue problem (14), (3) a straightforward discussion of the conditions for the existence of the bound states, (4) a unique approach to all analytically solvable problems of this kind leading to the matrix (21) in which only f_m and g_m appear. In this way, a common algebraic representation for all these problems has been found.

As the first application of our method, known results for the anharmonic oscillators have been critically recalculated and some new results have been obtained. It has been shown that the analytic solution is possible only if $2M = 4k+2$, where k is an integer. For the sextic $(k=1)$ and decadic $(k=2)$ oscillators a few new solutions for the asymmetric potential V have been given.

Another interesting problem is the generalized Morse oscillator, which is of interest in molecular physics. In contrast to the anharmonic oscillators, the analytic solutions exist for any $2M$. We have discussed analytic solutions for the quadratic, quartic, sextic, and higher-order oscillators. New results have been found for the quartic and higher-order generalized Morse oscillators. For the quartic oscillator, analytic solutions for the multiplets $n=0,1$ and $n=2$ have been discussed. The transition from the quartic Morse oscillator to the quartic anharmonic oscillator has also been made, confirming our previous conclusions. For the sextic oscillator, general formulas for g_m and the multiplet $n=0$ have been investigated.

Our method is applicable to any problem with the potential V and function f satisfying assumptions given above. Generalization to more dimensions is also possible.

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APPENDIX

It is interesting to notice that the case of the quadratic Morse potential, Eq. (30), can be treated using the algebraic methods. This was recognized by many authors and it is described, for example, in [22]. However, it is convenient for experimental purposes (see, e.g., [23], p. 8) to consider the potential in the form (32). For the algebraic approaches, we refer to the paper [25], namely, to Eq. (45). If we put $n=1$, $B=V_2$, $D=V_1+2V_2$, $b_0=1/2+\sqrt{V_2-V_1-E}$ and $\xi=V_1+V_2-E-1/4$ into Eqs. (42) and (52) of [24], the formula (33) is obtained.

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