PHYSICAL REVIEW A, VOLUME 63, 022107

WKB approach to calculating the lifetime of quasistationary states: Harmonic oscillator in a polynomial perturbation

J. Zamastil, J. Čížek, and L. Skála
Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Prague 2, Czech Republic and Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
(Received 12 July 2000; published 16 January 2001)

A simple and straightforward WKB approach to calculating the lifetime of quasistationary states in spherically symmetric potential wells is suggested. Using this approach, a general formula for the imaginary part of the energy for potentials of the form $V(x)=P(x)-\mu Q(x)$—where $P(x)$ is the radial part of the potential for a spherically symmetric harmonic oscillator and $Q(x)$ is an even polynomial—is derived. Using this formula, the usual tedious procedure of the explicit asymptotic matching of the WKB and perturbative wave functions is avoided, and calculations are substantially simplified. The leading term and a few corrections of the series for the imaginary part of the energy and the related lifetime are analytically calculated.

DOI: 10.1103/PhysRevA.63.022107 PACS number(s): 03.65.Sq

I. INTRODUCTION

In this paper, we are interested in calculating the lifetime of quasistationary states in the spherically symmetric D-dimensional potential wells

$$V(x)=P(x)-\mu Q(x),0<\mu\ll 1, \tag{1}$$

where

$$P(x)=\alpha x^{-2}+x^2 \tag{2}$$

is the radial part of the potential for the spherically symmetric harmonic oscillator,

$$Q(x)=\sum_{i=0}^{m-1} a_i x^{2(m-i)},m\geq 2 \tag{3}$$

is an even polynomial, and the coefficient α is equal to [1]

$$\alpha=l(l+D-2)+\frac{(D-1)(D-3)}{4}. \tag{4}$$

Here l denotes the orbital quantum number and a_i are real coefficients, $a_0=1$. This problem was studied from the point of view of quantum field theory [2–5] as well as a theory concerning a nonrelativistic hydrogen atom in a constant electric field [6–9].

Since the coupling constant μ is supposed to be small, the potential has one minimum near the origin, and goes to minus infinity for large x. From a physical point of view, this corresponds to the situation where the particle is localized inside the potential barrier and there is a small probability of its escaping through the barrier to infinity. Therefore, the energy $E=\mathrm{Re}\,E+i\,\mathrm{Im}\,E$ has a small imaginary part $\mathrm{Im}\,E<0$.

The WKB method can be used to calculate $\mathrm{Im}\,E$ and the related lifetime as a series in μ. These calculations were performed in a number of papers [2,4–12]. One disadvantage of the techniques used in these papers is that the correspondence of the terms of the WKB approximation to the terms of the series for $\mathrm{Im}\,E$ is not clear.

From a technical point of view, the WKB calculation of $\mathrm{Im}\,E$ is difficult. First, the calculation of the WKB wave function can easily be performed only at the leading order of the WKB method. Further, it is well known that the WKB method fails in the vicinity of the classical turning points. To avoid this difficulty, either the potential in the vicinity of the turning points is approximated by a straight line and the solution of the Schrödinger equation for this potential, the Airy function, is asymptotically matched to the WKB solution (see, e.g., Refs. [2,13]) or the WKB method is reformulated in such a way that it can directly be used at the turning points (the so-called Langer formulation; see, e.g., Refs. [7–10,13]). However, both these closely related methods lead to very long and tedious calculations. It was shown in Refs. [2,11] that this difficulty can be avoided in the leading order of the WKB method by taking the integral for the leading order in the complex plane. However, this calculation has never been extended to higher orders. Finally, it is necessary to perform an explicit asymptotic matching of the WKB and perturbative wave functions in an overlap region of mutual validity of both approximations. This is a rather tedious procedure [2,6–12], the complexity of which increases with increasing orders of the series for $\mathrm{Im}\,E$.

These difficulties explain the fact that no general WKB formula for $\mathrm{Im}\,E$ is known. Also, the WKB calculations in most cases were restricted to the leading order [6,9,11,12]. Corrections to the leading order were calculated in special cases only [2,7,8,10], mostly by means of numerical analysis [2,3,7,11,14].

WKB calculations were also performed via the path-integral approach [4,5,15]. However, calculations within this approach are restricted to the ground-state energy, and going beyond the leading term is extremely difficult [5].

The aim of this paper, which is a full version of the Ref. [16], is to suggest a WKB approach overcoming the problems mentioned above. First, a simple formulation of the WKB method for potentials (1) is suggested. This formulation not only substantially simplifies the calculation of the

WKB wave function in leading and higher orders, but also provides an interesting viewpoint on the semiclassical approximation in quantum mechanics. Second, a simple and general formula for calculating $\operatorname{Im} E$ for spherically symmetric potential wells of the form of Eq. (1) is derived. Using this formula, the usual difficult procedure of the explicit asymptotic matching of the WKB and perturbative wave functions is avoided, and calculations are substantially simplified. Moreover, this formula can be used regardless of the divergence of the WKB method at the turning points. The leading term and a few corrections of the series for $\operatorname{Im} E$ for potential (1) are calculated analytically. The large-order behavior of the series for the real part of the energy is calculated by means of the dispersion relation between the real and imaginary parts of the energy [2–4,7–11,15,14,17–19]. Our formulas include results known from previous papers as special cases.

The paper is organized as follows. In Sec. II, we reformulate the WKB method and derive a general formula for the imaginary part of the energy $\operatorname{Im} E$ of the quasistationary states in the potential wells [Eq. (1)]. In Sec. III, the leading order of the series for $\operatorname{Im} E$ is obtained. In Secs. IV, V, and VI, first-, second-, and third-order corrections to the leading order are found.

II. GENERAL THEORY

In this section, a WKB approach to calculating quasistationary states in the potential wells (1) is formulated. We are interested in calculating the lifetime of the quasistationary states of the Schrödinger equation

$$[-d^2/dx^2+V(x)]\psi(x)=E\psi(x), \tag{5}$$

with the potential $V(x)$ given by Eq. (1).

Writing the time-dependent wave function $\psi(x,t)$ in the form

$$\psi(x,t)=e^{-i(\operatorname{Re} E+i\operatorname{Im} E)t}\psi(x), \tag{6}$$

we obtain the time dependence of the probability density in the form

$$|\psi(x,t)|^2=e^{-t/\tau}|\psi(x)|^2, \tag{7}$$

where the lifetime τ is equal to

$$\tau=-\frac{1}{2\operatorname{Im} E}. \tag{8}$$

A. Probability current formula

To calculate $\operatorname{Im} E$, we multiply Eq. (5) by $\psi^*(x)$, and integrate it from 0 to infinity. Further, we take the complex conjugate of Eq. (5), multiply it by $\psi(x)$, and integrate from 0 to infinity. Taking the difference of the resulting two equations and integrating by parts we obtain (see, e.g., Ref. [2])

$$\operatorname{Im} E=\frac{\lim_{x\to\infty} J(x)}{\int_0^\infty |\psi(x')|^2 dx'}, \tag{9}$$

where

$$J(x)=\frac{1}{2i}\left[\psi(x)\frac{d}{dx}\psi^*(x)-\psi^*(x)\frac{d}{dx}\psi(x)\right] \tag{10}$$

is the probability current. Here we used the fact that the probability current at the origin is equal to zero:

$$J(x=0)=0. \tag{11}$$

Since Eq. (5) is solved in the interval $x\in(0,\infty)$, this condition is obeyed for the problem considered here. Equation (9) has a transparent physical meaning: the decrease of the probability to find a particle inside the potential barrier per time unit is proportional to the outgoing probability current.

B. RSPT approximation to the wave function

The wave function $\psi(x)$ for small x inside the potential barrier is calculated in the nth order of the Rayleigh-Schrödinger perturbation theory (RSPT)

$$\psi_{RSPT}^{(n)}(x)=\psi_0(x)+\mu\psi_1(x)+\mu^2\psi_2(x)+\cdots+\mu^n\psi_n(x). \tag{12}$$

The wave function $\psi_{RSPT}^{(n)}(x)$ is normalized to a constant discussed below.

The RSPT approximation of the wave function provides a good approximation to the exact wave function near the origin (for x satisfying the inequality $x^2\gg\mu x^{2m}$, i.e. for $x^2\ll\mu^{-1/(m-1)}$).

C. Norm of the wave function

The integral $\int_0^x|\psi(x')|^2dx'$ in the denominator of Eq. (9) can be calculated as follows [2,7,10]. Since the dominant contribution to the norm of the quasistationary wave function is given by small x, we replace $\psi(x)$ in this integral by $\psi_{RSPT}^{(n)}(x)$. Thus we obtain

$$\int_0^\infty|\psi(x')|^2dx'=\int_0^\infty|\psi_{RSPT}^{(n)}(x')|^2dx'. \tag{13}$$

Using $\psi_{RSPT}^{(n)}(x)$ in the form of Eq. (12) and assuming real $\psi_n(x)$, we can write

$$\int_0^\infty|\psi_{RSPT}^{(n)}(x')|^2dx'=\sum_{i=0}^{n} W_i\mu^i, \tag{14}$$

where

$$W_0=\int_0^\infty\psi_0(x)^2dx, \tag{15}$$

$$W_1=\frac{2\int_0^\infty \psi_1(x)\psi_0(x)dx}{W_0}, \tag{16}$$

$$W_2=\frac{2\int_0^\infty \psi_2(x)\psi_0(x)dx+\int_0^\infty \psi_1(x)^2dx}{W_0} \tag{17}$$

and so on.

D. WKB approximation to the wave function

The probability current at infinity is calculated by means of the WKB wave function $\psi_{WKB}^{(n)}(x)$. To calculate this function we suggest a simple approach. The dominant contribution to the probability current at infinity comes from the classically forbidden region $x^2\approx\mu x^{2m}$, where $V(x)>\mathrm{Re}\,E$ [2,6,11,12]. Therefore, we perform the substitution $x=\mu^{-1/[2(m-1)]}u$ in Eq. (5) to make the terms x^2 and μx^{2m} of the same order in μ. We obtain the equation

$$\mu^{2/(m-1)}\frac{d^2}{du^2}\psi=[u^2-u^{2m}-(E_0+a_1u^{2(m-1)})\mu^{1/(m-1)}+\cdots]\psi, \tag{18}$$

where the RSPT expansion of the real part of the energy

$$\mathrm{Re}\,E=E_0+\mu E_1+\mu^2E_2+\cdots \tag{19}$$

was used. Searching for the solution in the form of the WKB expansion,

$$\psi_{WKB}^{(n)}(u)=\exp\left[\frac{1}{\mu^{1/(m-1)}}\sum_{i=0}^{1+n(m-1)}\times[S_i(u)-A_i]\mu^{i/(m-1)}\right], \tag{20}$$

and comparing the terms of the same powers of $\mu^{1/(m-1)}$, we obtain equations from which the $S_i(u)=S_i(\mu^{1/[2(m-1)]}x)$ terms can be easily calculated. Here A_i are normalization constants discussed below.

The first two terms of the expansion are equal to

$$S_0(u)=-\int u[1-u^{2(m-1)}]^{1/2}du \tag{21}$$

and

$$S_1(u)=-\frac{1}{4}\ln(u^2-u^{2m})+\int\frac{E_0+a_1u^{2(m-1)}}{2u[1-u^{2(m-1)}]^{1/2}}du. \tag{22}$$

Here integration constants are included into the normalization constants A_i. We took the minus sign in Eq. (21) to obtain an exponentially decaying solution corresponding to the particle moving through the potential barrier to infinity. The $S_1(u)$ term can be calculated analytically:

$$S_1(u)=-\frac{1}{4}\ln(u^2-u^{2m})-\frac{E_0}{4(m-1)}\ln\frac{1+[1-u^{2(m-1)}]^{1/2}}{1-[1-u^{2(m-1)}]^{1/2}}-\frac{a_1}{2}\frac{[1-u^{2(m-1)}]^{1/2}}{m-1}. \tag{23}$$

The form of our WKB wave function $\psi_{WKB}^{(n)}$ is the same as that obtained by taking the first $2+n(m-1)$ terms of the usual semiclassical expansion, and expanding them up to the nth order of μ. However, our approach is much simpler and more straightforward.

The WKB method yields a good approximation to the wave function for x satisfying inequality $x^2\gg\mathrm{Re}\,E$. The norm of the wave function is calculated from the RSPT wave function and the probability current at infinity from the WKB wave function. To obtain meaningful results we have to guarantee that these function have the same normalization. This is done via an asymptotic matching of these functions [2,6–13].

E. Asymptotic matching

Now we suggest a simple method to asymptotically match the functions $\psi_{RSPT}^{(n)}(x)$ and $\psi_{WKB}^{(n)}(x)$. In an overlap region of mutual validity of both approximations the logarithm of the RSPT and WKB functions [Eqs. (12) and (20)], expanded up to the nth order of μ, must (for large x) be asymptotically equal,

$$\sum_{i=0}^{1+n(m-1)}S_i(\mu^{1/[2(m-1)]}x)\mu^{(i-1)/(m-1)}=\ln\psi_{RSPT}^{(n)}(x)+\sum_{i=0}^{1+n(m-1)}A_i\mu^{(i-1)/(m-1)}+O(1/x^2). \tag{24}$$

The crucial point of our approach is that the right-hand side of Eq. (24) can be considered as the asymptotic expansion of the left-hand side. Generally, this expansion contains x-dependent terms, constant terms, and terms negligible in the overlap region. Henceforth, we normalize $\psi_{RSPT}^{(n)}(x)$ in such a way that the asymptotic expansion of $\ln\psi_{RSPT}^{(n)}(x)$ does not contain a constant additive term in any order of μ. Then, it follows from Eq. (24) that $\Sigma A_i\mu^{(i-1)/(m-1)}$ *is a constant term of the asymptotic expansion of* $\Sigma S_i(\mu^{1/[2(m-1)]}x)\mu^{(i-1)/(m-1)}$ *in the overlap region.* It follows from the requirement that the asymptotic matching can be performed for an arbitrary μ that A_i *is a constant term of the asymptotic expansion of* $S_i(\mu^{1/[2(m-1)]}x)$ *in the overlap region*.

It is seen from the way in which the both approximations are constructed, that the overlap region is given by inequalities $x^2\gg\mu x^{2m}$ and $x^2\gg\mathrm{Re}\,E$ i.e. for $\mathrm{Re}\,E\ll x^2\ll\mu^{-1/(m-1)}$. This interval exists if we take a sufficiently small $\mathrm{Re}\,E$ for fixed μ, or a sufficiently small μ for fixed $\mathrm{Re}\,E$.

F. Asymptotics of the wave function and the probability current at infinity

Taking large $u\to\infty$ in Eqs. (21) and (23), we obtain the asymptotics of the WKB wave function for $x\to\infty$ identical with the asymptotics of the exact wave function

$$\psi_{WKB}^{(n)}(x\to\infty)=\exp\left[\sum_{i=0}^{1+n(m-1)}(C_i-A_i)\mu^{(i-1)/(m-1)}\right]\times\frac{\exp\{i\mu^{1/2}x^{m+1}/(m+1)[1+O(1/x^2)]\}}{x^{m/2}}. \tag{25}$$

Here the outgoing wave is given by the positive sign in the second exponential which leads to $\mathrm{Im}\,E<0$. Analogously to A_i terms, C_i *denotes the constant term of the asymptotic expansion of* $S_i(\mu^{1/[2(m-1)]}x)$ *for x going to infinity*. Using Eq. (25) in Eq. (10), we obtain the formula for the probability current at infinity:

$$\lim_{x\to\infty}J(x)=-\mu^{1/2}\exp\left[2\,\mathrm{Re}\sum_{i=0}^{1+n(m-1)}(C_i-A_i)\mu^{(i-1)/(m-1)}\right]. \tag{26}$$

G. Formula for the imaginary part of the energy

Inserting Eqs. (13), (14) and (26) into Eq. (9), we obtain a simple final formula for the imaginary part of the energy

$$\mathrm{Im}\,E=-\mu^{1/2}\frac{\exp\left[2\,\mathrm{Re}\sum_{i=0}^{1+n(m-1)}(C_i-A_i)\mu^{(i-1)/(m-1)}\right]}{\sum_{i=0}^{n}W_i\mu^i}. \tag{27}$$

The lifetime τ can be calculated via Eq. (8).

Formula (27) shows that to calculate the imaginary part of the energy it is sufficient to calculate the difference of the real parts of the constant terms of the asymptotic expansion of $S_i(\mu^{1/[2(m-1)]}x)$ in the overlap region and at infinity, and to calculate the norm of the RSPT wave function.

Our WKB approximation cannot be used at the point $u=1$; see Eq. (23). However, since we need to calculate only the constant terms of the asymptotic expansion in the overlap region and at infinity, Eq. (27) can be used regardless of the divergence of the WKB approximation at this point.

H. Calculation of the constant terms

In this subsection, we discuss calculation of C_i and A_i terms in Eq. (27). We distinguish two cases. If the integrals giving $S_i(u)$ terms can be calculated analytically, then the A_i and C_i terms can be obtained directly from the definition. If it is not the case, the difference of the real parts of C_i and A_i can be calculated via an integral formula derived below.

1. Direct calculation

For the potentials with $m=2$ and $m=3$ all the integrals in $S_i(u)$ terms can be calculated analytically. Except for the logarithmically dependent $S_1(u)$ term, the expansion of $S_i(u)$ near $u=0$ leads to a power series in u^2. This expansion generally contains terms proportional to the positive powers of u^2, constant term and terms proportional to the negative powers of u^2.

Since $S_i(\mu^{1/[2(m-1)]}x)$ depend on μ and x via the variable $u=\mu^{1/[2(m-1)]}x$ and all the terms A_i, $i\neq 1$ do not depend on x and μ, it is seen that *the constants* A_i *for* $i\neq 1$ *are equal to the constant terms of the expansion of* $S_i(u)$ *near* $u=0$ *and* $i\neq 1$. Similarly, C_i *terms for* $i\neq 1$ *are equal to the constant terms of the asymptotic expansion of* $S_i(u)$ *for u going to infinity*.

The $S_1(u)$ term can be calculated analytically for all m; see Eq. (23). The expansion of this term near $u=0$ gives a logarithmically dependent term, a constant term, and terms proportional to the positive powers of u^2. To calculate the constant term of the asymptotic expansion of the logarithmic term, we return to the variable x,

$$S_1(\mu^{1/[2(m-1)]}x)=-\frac{1}{4}\ln x^2-\frac{\ln\mu}{4(m-1)}-\left[\frac{E_0}{4(m-1)}\ln\frac{4}{u^{2(m-1)}}+\frac{a_1}{2(m-1)}+O(u^2)\right]^{x\mu^{1/[2(m-1)]}}, \tag{28}$$

and obtain

$$A_1=-\frac{\ln\mu}{4(m-1)}-\frac{E_0}{4(m-1)}\ln\frac{4}{\mu}-\frac{a_1}{2(m-1)}. \tag{29}$$

Similarly, the expansion of the real part of $S_1(u)$ at infinity is equal to

$$\mathrm{Re}\,S_1(x\to\infty)=\left[-\frac{\ln u^{2m}}{4}+O(1/u)\right]^{\mu^{1/[2(m-1)]}x}. \tag{30}$$

Hence, the real part of C_1 reads

$$\mathrm{Re}\;C_1=-\frac{\ln\mu^{m/(m-1)}}{4}. \tag{31}$$

2. Integral formula

For the potentials with $m>3$, the integrals in $S_i(u)$ terms cannot be calculated analytically. In this case, the terms in the numerator in Eq. (27) can be calculated as follows. We split $S_i(u)$ into two parts,

$$S_i(u)=\int b_i(u)du+D_i(u), \tag{32}$$

where the first term satisfies the condition

$$\int_0^1 b_i(u)du < \infty. \tag{33}$$

Since the b_i terms are real for $u<1$ and purely imaginary for $u>1$, we can write

$$\mathrm{Re}\,(C_i - A_i) = \int_0^1 b_i(u)du + \mathrm{Re}(V_i - P_i), \quad i \neq 1, \tag{34}$$

where P_i and V_i denote constant terms of the expansion of $D_i(u)$ near zero and infinity, respectively. If we are not able to calculate the integrals in $S_i(u)$ terms analytically, we integrate them by parts until the remaining integral satisfies condition (33).

I. Dispersion relation

The RSPT coefficients of the energy [Eq. (19)] for large n can be calculated from the dispersion relation

$$E_n = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,E(\mu)}{\mu^{n+1}} d\mu. \tag{35}$$

For the purposes of this paper, relation (35) can be used for the numerical verification of formula (27). The large-order E_n coefficients can be calculated numerically either via the difference equation method [2,20] or by means of the so(2,1) algebraic formulation of the perturbation theory [21]. Thus, using Eq. (35) and, for example, Thiele-Padé or Neville-Richardson extrapolation [13,22], the analytical formula [Eq. (27)] can be compared with the numerical values of the large-order perturbation coefficients E_n.

III. LEADING TERM

In this section, we calculate the leading term of the imaginary part of the energy [Eq. (27)]. The calculation can be separated into two steps: the RSPT part [the calculation of the denominator in Eq. (27)] and the WKB part [the calculation of the numerator in Eq. (27)].

A. RSPT part of the calculation

The function $\psi_0(x)$ is equal to the wave function of the multidimensional spherically symmetric harmonic oscillator

$$H_0\psi_0(x) = E_0\psi_0(x), \tag{36}$$

where

$$H_0 = -\frac{d^2}{dx^2} + \alpha x^{-2} + x^2. \tag{37}$$

Except for the normalization constant, this function is equal to [21]

$$\psi_0(x) \approx e^{-x^2/2} x^{2b_0 - 1/2} L_{N-b_0}^{(2b_0-1)}(x^2), \tag{38}$$

where $L_{N-b_0}^{(2b_0-1)}(x^2)$ denotes the associated Laguerre polynomial [23], N is the principal quantum number related to the unperturbed energy level by the equation

$$E_0 = 4N, \tag{39}$$

and the coefficient b_0 is equal to

$$b_0 = \frac{l}{2} + \frac{D}{4}. \tag{40}$$

The normalization constant is found from the requirement that the asymptotic expansion of $\ln \psi_0(x)$ does not contain a constant additive term. The asymptotics of the polynomials $L_{N-b_0}^{(2b_0-1)}(x^2)$ is equal to [23]

$$L_{N-b_0}^{(2b_0-1)}(x^2) = \frac{(-1)^{(N-b_0)}}{(N-b_0)!} x^{2(N-b_0)}[1 + O(1/x^2)]. \tag{41}$$

Hence the normalization constant has to be equal to $(-1)^{(N-b_0)}(N-b_0)!$. Thus, the unperturbed wave function reads

$$\psi_0(x) = \langle x|N\rangle = (-1)^{(N-b_0)}(N-b_0)!\,e^{-x^2/2} x^{2b_0-1/2} L_{N-b_0}^{(2b_0-1)}(x^2) \tag{42}$$

and the norm of this function is [23]

$$W_0 = \frac{\Gamma(N-b_0+1)\Gamma(N+b_0)}{2}. \tag{43}$$

B. WKB part of the calculation

Since the integral in Eq. (21) satisfies condition (33), the difference $C_0 - A_0$ can be calculated as follows:

$$\mathrm{Re}\,(C_0 - A_0) = -\int_0^1 u[1-u^{2(m-1)}]^{1/2} du = -\frac{1}{2(m-1)} \frac{\Gamma\left(\frac{1}{m-1}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{m-1}\right)}. \tag{44}$$

Here we used the formula for the Γ function [23]:

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \; \mathrm{Re}\,x>0, \mathrm{Re}\,y>0. \tag{45}$$

Taking the difference of the real parts of Eqs. (29) and (31), we obtain

$$\mathrm{Re}\,(C_1 - A_1) = -\ln \mu^{1/4} + \frac{E_0}{4(m-1)} \ln\frac{4}{\mu} + \frac{a_1}{2(m-1)}. \tag{46}$$

C. Final result

Inserting Eqs. (43), (44), and (46) into Eq. (27), wec an write the series for $\mathrm{Im}\,E$,

$$\mathrm{Im}\,E=-\frac{2e^{a_1/(m-1)}}{(N-b_0)!\Gamma(N+b_0)}\left(\frac{4}{\mu}\right)^{2N/(m-1)}\times e^{-\mu^{-1/(m-1)}d}[1+R_1\mu^{1/(m-1)}+R_2\mu^{2/(m-1)}+\cdots], \tag{47}$$

where

$$d=\frac{\Gamma\left(\frac{1}{m-1}\right)\Gamma\left(\frac{3}{2}\right)}{(m-1)\Gamma\left(\frac{3m-1}{2(m-1)}\right)}. \tag{48}$$

Coefficients R_1, R_2, ..., which are neglected in leading order, are calculated in the following sections.

Substituting Eq. (47) into Eq. (35), we obtain the large-order behavior of the RSPT coefficients

$$E_n=-\frac{4^{2N/(m-1)}2(m-1)}{\pi(N-b_0)!\Gamma(N+b_0)}d^{-2N-n(m-1)}\Gamma[(m-1)n+2N]\times\left[1+\frac{R_1d}{(m-1)n+2N-1}+\frac{R_2d^2}{[(m-1)n+2N-1][(m-1)n+2N-2]}+\cdots\right]. \tag{49}$$

The results known from previous papers can be obtained as special cases of Eqs. (47) and (49). The even and odd states of the one-dimensional potentials can be obtained by setting $l=0$ and 1, respectively. Thus, for $D=1$, $l=0$ and $l=1$, Eqs. (47) and (49) yield the formulas given in Ref. [2]. For $l=0$, $K=0$, and $a_i=0$ for $i>0$, the formula given in Ref. [4] can be obtained. It is well known that the problem of the hydrogen atom in the constant electric field can be transformed to the problem of the two-dimensional quartic oscillator [6–8]. Putting $E=4\mu_2$, $\mu=4f$, $l=m_p$, $K=n_2$, $D=2$, $a=0$, and $m=2$, where m_p is the magnetic quantum number, n_2 is the second parabolic quantum number, μ_2 is the separation constant of the Schrödinger equation for the hydrogen atom in the parabolic coordinates, and f is the intensity of the electric field, from Eq. (47) we obtain the leading term of the formula for the lifetime of the quasistationary states of the hydrogen atom in a constant electric field [Eq. (44) in Ref. [7]].

IV. FIRST-ORDER CALCULATION

In this section we calculate the coefficient R_1 in Eq. (47).

A. RSPT part of the calculation

Since the first-order correction to the norm of the wave function is proportional to μ [see Eq. (27)], it has to be taken into account for the potentials with $m=2$ only. Calculating $\psi_{RSPT}^{(1)}(x)$, we have to keep in mind that this function must be normalized in such a way that the asymptotic expansion of $\ln\psi_{RSPT}^{(1)}$ in the first order of μ does not contain a constant additive term. Assuming $\psi_{RSPT}^{(1)}$ in the form of Eq. (12), its logarithm in the first order of μ is equal to

$$\ln\psi_{RSPT}^{(1)}(x)=\ln\left[\psi_0(x)\left(1+\mu\frac{\psi_1(x)}{\psi_0(x)}\right)\right]=\ln\psi_0(x)+\mu\frac{\psi_1(x)}{\psi_0(x)}+O(\mu^2). \tag{50}$$

This means that $\psi_1(x)$ has to be normalized in such a way that the asymptotic expansion of $\psi_1(x)/\psi_0(x)$ does not contain a constant term. Therefore, to obtain a first-order correction to the norm of the wave function, we have to proceed as follows. First, we calculate $\psi_1(x)$ from the RSPT. Second, we normalize this function as described above. Finally, we insert this function into Eq. (16).

1. Calculation of $\psi_1(x)$

To calculate the integrals needed in the perturbation theory, it is convenient to introduce the inner product

$$\langle N|M\rangle=\int_0^\infty\langle N|x\rangle\langle x|M\rangle dx. \tag{51}$$

With this definition, the eigenvectors $|N\rangle$ given by Eq. (42) form an orthogonal set [23]:

$$\langle N|M\rangle=\delta_{N,M}\frac{\Gamma(N-b_0+1)\Gamma(N+b_0)}{2}. \tag{52}$$

As usual in the perturbation theory, we expand the energy and wave function in the series in the coupling constant μ,

$$E=E_0+\mu E_1+\cdots, \tag{53}$$

$$|\psi\rangle=|N\rangle+\mu|\psi_1\rangle+\cdots, \tag{54}$$

and the perturbation function into the unperturbed basis set

$$|\psi_1\rangle=\sum_{n=N-2}^{N+2}g_{n-N}^{(1)}|n\rangle. \tag{55}$$

Here the eigenvectors $|n\rangle$ are given by Eq. (42) for $N=n$. Inserting these expansions into Eq. (5), in the first order of μ we obtain

$$\sum_{n=N-2}^{N+2} g_{n-N}^{(1)}(H_0-4N)|n\rangle=(x^4+a_1x^2+E_1)|N\rangle. \quad (56)$$

To calculate the coefficients $g_{n-N}^{(1)}$ and E_1, we have to know the matrix elements of x^2 and x^4. The calculation of these matrix elements is greatly simplified by introducing the ladder operators [21]

$$T_{\pm}=\frac{1}{4}\left(-\frac{d^2}{dx^2}+\alpha x^{-2}-x^2\right)\pm\frac{1}{2}\left(x\frac{d}{dx}+\frac{1}{2}\right). \quad (57)$$

Applying these operators to the eigenvectors [Eq. (42)], we obtain

$$T_+|N\rangle=-|N+1\rangle \quad (58)$$

and

$$T_-|N\rangle=-(N+b_0-1)(N-b_0)|N-1\rangle. \quad (59)$$

Using Eqs. (37) and (57) we express x^2 via the operators H_0, T_+, and T_-:

$$x^2=\frac{H_0}{2}-(T_++T_-). \quad (60)$$

It follows from Eqs. (36), (39), (58), and (59) that

$$x^2|N\rangle=|N+1\rangle+2N|N\rangle+(N+b_0-1)(N-b_0)|N-1\rangle \quad (61)$$

and

$$\begin{aligned}x^4|N\rangle=&|N+2\rangle+(4N+2)|N+1\rangle+\left(6N^2-\frac{\alpha}{2}+\frac{3}{8}\right)|N\rangle\\&+(4N-2)(N+b_0-1)(N-b_0)|N-1\rangle\\&+(N+b_0-1)(N+b_0-2)(N-b_0)(N-b_0-1)\\&\times|N-2\rangle.\end{aligned} \quad (62)$$

Here the equality

$$b_0(b_0-1)=\frac{\alpha}{4}-\frac{3}{16} \quad (63)$$

following from Eqs. (4) and (40) was used.

By inserting Eqs. (61) and (62) into Eq. (56), and using the orthogonality relation [Eq. (52)], we obtain equations for $g_{n-N}^{(1)}$ and E_1. After some manipulation we obtain

$$g_2^{(1)}=\frac{1}{8}, \quad (64)$$

$$g_1^{(1)}=N+\frac{1}{2}+\frac{a_1}{4}, \quad (65)$$

$$E_1=-\left(6N^2-\frac{\alpha}{2}+\frac{3}{8}+2Na_1\right). \quad (66)$$

We do not write the values of the coefficients $g_{-1}^{(1)}$ and $g_{-2}^{(1)}$ which will not be needed here.

The perturbation theory does not give any value of the coefficient $g_0^{(1)}$. This coefficient can be used to normalize $|\psi_1\rangle$.

2. Normalization of $\psi_1(x)$

The condition that the asymptotic expansion of $\psi_1(x)/\psi_0(x)$ does not contain a constant term reads

$$\frac{\psi_1(x)}{\psi_0(x)}=d_2^{(1)}x^4+d_1^{(1)}x^2+O(1/x^2). \quad (67)$$

This equation contains three unknowns $d_2^{(1)}$, $d_1^{(1)}$, and $g_0^{(1)}$. These unknowns can be found by comparing the terms proportional to the same non-negative powers of x^2.

To compare these terms we use the asymptotic form of eigenfunction (42) [23],

$$\begin{aligned}\langle x|N\rangle=&e^{-x^2/2}x^{2N-1/2}[1+f_1(N)x^{-2}+f_2(N)x^{-4}+O(x^{-6})\\&+\cdots],\end{aligned} \quad (68)$$

where the coefficients $f_1(N)$ and $f_2(N)$ are given by the recurrence relations between the coefficients of the Laguerre polynomials [23]:

$$f_1(N)=-(N+b_0-1)(N-b_0) \quad (69)$$

and

$$f_2(N)=\frac{(N+b_0-1)(N+b_0-2)(N-b_0)(N-b_0-1)}{2}. \quad (70)$$

By inserting expansions (55) and (68) for $\psi_1(x)$ and $\langle x|n\rangle$ into Eq. (67), and multiplying this equation by $\psi_0(x)$, we find

$$\begin{aligned}&g_2^{(1)}[x^4+f_1(N+2)x^2+f_2(N+2)]+g_1^{(1)}\\&\quad\times[x^2+f_1(N+1)]+g_0^{(1)}+O(1/x^2)\\&=[1+f_1(N)x^{-2}+f_2(N)x^{-4}][d_2^{(1)}x^4+d_1^{(1)}x^2]\\&\quad+O(1/x^2).\end{aligned} \quad (71)$$

By inserting Eqs. (64), (65), (69), and (70) into this equation and comparing the terms proportional to the same powers of x^2, we obtain a system of three linear equations for three unknowns that can be easily solved:

$$d_2^{(1)}=\frac{1}{8}, \quad (72)$$

$$d_1^{(1)}=\frac{N+2}{8}, \quad (73)$$

and

$$g_0^{(1)}=\frac{7}{4}N^2+\frac{N}{2}(1+a_1)-\frac{\alpha}{16}+\frac{3}{64}. \tag{74}$$

The last equation gives the normalization of $\psi_1(x)$.

3. Correction to the norm of the wave function

Using the inner product [Eq. (51)], Eq. (16) can be written in the form

$$W_1=\frac{2\langle N|\psi_1\rangle}{\langle N|N\rangle}. \tag{75}$$

Taking into account Eqs. (52) and (55), we obtain

$$W_1=2g_0^{(1)}. \tag{76}$$

By inserting Eq. (74) into the last equation, we obtain the final expression for the first-order correction to the norm of the wave function:

$$W_1=\frac{7}{2}N^2+N(1+a_1)-\frac{\alpha}{8}+\frac{3}{32}. \tag{77}$$

B. WKB part of the calculation

1. Potentials with $m=2$

Using Eqs. (19), (20), and (66) in Eq. (18), and comparing the terms proportional to the second order of μ, we obtain

$$S_2(u)=\int\frac{-\frac{d^2S_1(u)}{du^2}-\left(\frac{dS_1(u)}{du}\right)^2+\alpha u^{-2}-\left(6N^2-\frac{\alpha}{2}+\frac{3}{8}+2Na_1\right)}{2\frac{dS_0(u)}{du}}du. \tag{78}$$

The integrals in Eq. (78) can be calculated analytically. Expanding the resulting expression for $S_2(u)$ in u^2 near $u=0$, we obtain a term proportional to u^{-2}, a constant term, and terms proportional to positive powers of u^2. The constant term A_2 is equal to

$$A_2=-\frac{\alpha}{8}+\frac{5N^2}{2}+\frac{a_1^2}{8}+N(1+a_1)+\frac{a_1}{4}+\frac{29}{96}. \tag{79}$$

The constant term C_2 is equal to zero:

$$\operatorname{Re}C_2=0. \tag{80}$$

2. Potentials with $m>2$

Using Eq. (20) in Eq. (18) and comparing the terms proportional to $\mu^{2/(m-1)}$, we obtain

$$S_2(u)=\int\frac{-\frac{d^2S_1(u)}{du^2}-\left(\frac{dS_1(u)}{du}\right)^2+\alpha u^{-2}-a_2u^{2(m-2)}}{2\frac{dS_0(u)}{du}}du. \tag{81}$$

To calculate the constant terms we use Eq. (34). After a long but straightforward calculation, we obtain

$$\operatorname{Re}(C_2-A_2)=\left\{\left[-\frac{\alpha(m-1)}{4}+\left(N^2-\frac{1}{8}\right)(m+1)+\frac{m^2}{12}+\frac{m}{16}+\frac{5}{48}+Na_1\right]\frac{3-m}{m-1}-\frac{a_1^2(m-2)}{4(m-1)}+\frac{a_2}{2}\right\}I, \tag{82}$$

where the integral

$$I=\int_0^1\frac{u^{2m-5}du}{(1-u^{2(m-1)})^{1/2}}=\frac{1}{2(m-1)}\frac{\Gamma\left(\frac{m-2}{m-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{m-2}{m-1}\right)} \tag{83}$$

was calculated by means of Eq. (45). In this case, only the first term in Eq. (32) contributes to the real part of the constant terms of $S_2(u)$ near zero and infinity. The real part of the constant terms of $D_2(u)$ is equal to zero:

$$\operatorname{Re}(V_2-P_2)=0. \tag{84}$$

C. Final results

Inserting Eqs. (77), (79), and (80) into Eq. (27) for $m=2$ we obtain

$$R_1=\frac{3\alpha}{8}-3N(1+a_1)-\frac{17N^2}{2}-\frac{67}{96}-\frac{a_1}{2}-\frac{a_1^2}{4}. \tag{85}$$

Substituting Eq. (82) into Eq. (27), for $m>2$ we obtain

$$R_1=\left\{\left[-\frac{\alpha(m-1)}{4}+\left(N^2-\frac{1}{8}\right)(m+1)+\frac{m^2}{12}+\frac{m}{16}+\frac{5}{48}+Na_1\right](3-m)-\frac{a_1^2(m-2)}{4}+\frac{a_2(m-1)}{2}\right\}\frac{1}{(m-1)^2}\frac{\Gamma\left(\frac{m-2}{m-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3m-5}{2(m-1)}\right)}. \tag{86}$$

These formulas contain formulas known from previous papers as special cases. For $D=1$, $l=0$, $l=1$, and $a_1=-3$, Eq. (85) yields the formula found in Ref. [2]. For $l=0$, $K=0$, and $a_i=0$ for $i>0$, the formula conjectured in Ref. [3] from the numerical analysis is obtained. Our result is the analytic derivation of this conjecture. The coefficient R_1 for the hydrogen atom in the constant electric field [7,8] can be obtained from Eq. (85) in a similar way as the leading term of this problem from Eq. (47). Setting $K=0$, $l=0$, $D=2$ and 4, $m=3,\dots,\ 6$, and $a_i=0$ for $i>0$ in Eq. (86) after some manipulation we obtain the numerical values given in Table VII of Ref. [14].

V. SECOND-ORDER CALCULATION

The second order calculation is analogous to the first-order one. Here we only describe the calculation of the second-order correction to the norm of the wave function. The WKB part of the calculation does not require any special comments. We take into account the $S_3(u)$ term in Eq. (20), and calculate the constant terms near infinity and zero. Proceeding similarly to the previous section we first calculate the coefficients of the expansion

$$|\psi_2\rangle=\sum_{n=N-4}^{N+4} g_{n-N}^{(2)}|n\rangle \tag{87}$$

from the RSPT.

Second, the logarithm of $\psi_{RSPT}^{(2)}(x)$ equals in the second order of μ:

$$\ln \psi_{RSPT}^{(2)}(x)=\ln \psi_0(x)+\mu\frac{\psi_1(x)}{\psi_0(x)}+\mu^2\left[\frac{\psi_2(x)}{\psi_0(x)}-\frac{1}{2}\left(\frac{\psi_1(x)}{\psi_0(x)}\right)^2\right]+O(\mu^3). \tag{88}$$

Therefore, we require that the asymptotic expansion of the term in the square brackets does not contain a constant term:

$$\frac{\psi_2(x)}{\psi_0(x)}-\frac{1}{2}\left(\frac{\psi_1(x)}{\psi_0(x)}\right)^2=d_4^{(2)}x^8+d_3^{(2)}x^6+d_2^{(2)}x^4+d_1^{(2)}x^2+O(1/x^2). \tag{89}$$

This is the equation for five unknowns $d_4^{(2)}$, $d_3^{(2)}$, $d_2^{(2)}$, $d_1^{(2)}$, and $g_0^{(2)}$. On the left-hand side, we use the complete expansion [Eq. (55)] of $\psi_1(x)$, the coefficients $g_4^{(2)}$, $g_3^{(2)}$, $g_2^{(2)}$, and $g_1^{(2)}$ of the expansion of $\psi_2(x)$, and the coefficients $f_1(N)$, $f_2(N)$, $f_3(N)$, and $f_4(N)$ of expansion (68). Comparing the terms proportional to the same powers of x^2, we obtain a system of five linear equations for five unknowns.

After solving this problem and calculating the coefficient $g_0^{(2)}$, we calculate W_2 from Eq. (17),

$$W_2=\frac{2\langle\psi_2|N\rangle+\langle\psi_1|\psi_1\rangle}{\langle N|N\rangle}=2g_0^{(2)}+\sum_{n=N-2}^{N+2}(g_{n-N}^{(1)})^2\frac{\langle n|n\rangle}{\langle N|N\rangle}, \tag{90}$$

where the products $\langle n|n\rangle$ are given by Eq. (52) for $M=N=n$.

In this way, for $m=2$ and $a_1=0$ we also derived the second-order correction coefficient

$$R_2=\frac{1}{16}\left[578N^4-92N^3-\frac{442}{3}N^2-90N+18\xi^2+(-204N^2+100N+26)\xi-\frac{155}{18}\right], \tag{91}$$

where

$$\xi=\frac{\alpha}{4}-\frac{3}{16}. \tag{92}$$

For $D=2$, from Eq. (91) we obtain the result given in Ref. [7]. The formula conjectured in Ref. [14] can be obtained from Eq. (91) for $l=0$ and $K=0$.

By taking into account the S_3 term in Eq. (20) and the first-order correction to the norm of the wave function, for $m=3$ and $a_i=0$, for $i>0$, we obtain

$$R_2=-\frac{68}{3}N^3-15N^2+\left(-\frac{22}{3}+10\xi\right)N-1+3\xi. \tag{93}$$

By performing the WKB part of the calculation, for $m=4$ and $a_i=0$, for $i>0$, we obtain

$$\begin{aligned}R_2=&\frac{25}{18}\frac{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}{\pi}N^4+\frac{182}{243}\frac{\pi^{3/2}3^{1/2}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}N^3\\&-\frac{5}{18}\frac{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}{\pi}(6\xi-1)N^2-\frac{7}{27}\frac{\pi^{3/2}3^{1/2}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}\\&\times(2\xi-1)N+\frac{1}{72}\frac{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}{\pi}(1-12\xi+36\xi^2).\end{aligned} \tag{94}$$

For $K=0$, $l=0$, $D=2$, and $D=4$ this equation leads to numerical values given in Table VII of Ref. [14].

VI. THIRD-ORDER CALCULATION

By performing the WKB part of the calculation, for $m=3$ and $a_i=0$, for $i>0$, we obtain

$$R_3=0. \tag{95}$$

By taking into account the S_4 term in Eq. (20) and W_1, for $m=4$ and $a_i=0$, for $i>0$, we obtain

$$R_3=-\frac{125}{162}\frac{\Gamma\left(\frac{2}{3}\right)^3\Gamma\left(\frac{5}{6}\right)^3}{\pi^{3/2}}N^6-\frac{910}{729}\pi 3^{1/2}N^5$$
$$+\left[\frac{\Gamma\left(\frac{2}{3}\right)^3\Gamma\left(\frac{5}{6}\right)^3}{\pi^{3/2}}\left(-\frac{25}{108}+\frac{25}{18}\xi\right)-\frac{2587}{36}\right]N^4$$
$$-\left[3^{1/2}\pi\left(\frac{406}{729}-\frac{392}{243}\xi\right)+70\right]N^3$$
$$-\left[\frac{\Gamma\left(\frac{2}{3}\right)^3\Gamma\left(\frac{5}{6}\right)^3}{\pi^{3/2}}\frac{5}{216}(6\xi-1)^2+\frac{2495}{36}-\frac{301}{6}\xi\right]N^2$$
$$+\left[-3^{1/2}\pi\frac{7}{162}(6\xi-1)(2\xi-1)+30\xi-25\right]N$$
$$+\frac{\Gamma\left(\frac{2}{3}\right)^3\Gamma\left(\frac{5}{6}\right)^3}{\pi^{3/2}}\frac{1}{1296}(6\xi-1)^3-\frac{129}{40}+\frac{34}{3}\xi-\frac{41}{12}\xi^2. \quad (96)$$

VII. CONCLUSIONS

In this paper, we have used the WKB method to calculate the lifetime of quasistationary states in spherically symmetric potential wells of the form of Eq. (1). A straightforward method of calculating the WKB wave functions was suggested. Furthermore, we showed that the usual tedious procedure of explicit asymptotic matching of the WKB and RSPT wave functions can be avoided.

A general formula [Eq. (27)] for the lifetime was derived. In comparison with previous approaches, the use of this formula has the following advantages. First, to our knowledge, a general formula similar to Eq. (27) has not been known till now, and any problem had to be solved from the very beginning. Second, this formula offers a systematic way of performing calculations to high orders of the coupling constant μ. Finally, this equation uses only minimal information necessary for calculating $\mathrm{Im}\,E$.

The leading order of the WKB method and a few corrections were explicitly calculated, and analytical results obtained. Results known from the literature follow from our results as special cases. Our results were also verified numerically.

With only a slight modification, our method can be applied to the problem of a hydrogen atom in a spherically symmetric polynomial perturbation [11,14,21], and to a harmonic oscillator in cubic perturbation [24]. These problems will be discussed elsewhere.

Since the approach discussed in this paper is based on the scaling property of potential (1), this method seems to be restricted to polynomial potentials (or potentials that can be well approximated by polynomial potentials). On the other hand, this method is not restricted to separable multidimensional problems. The extension of our method to nonseparable multidimensional problems will be discussed in a forthcoming paper.

The RSPT and WKB calculations provide series for the real and imaginary parts of the energy [Eqs. (19) and (27)], respectively. The greater the number of terms of series (19) and (27) that is known, the greater the amount of detailed information about the quasistationary states is available. Since the method outlined in this paper offers a systematic and straightforward way to obtain this information from the Schrödinger equation, we believe that it is of considerable interest.

ACKNOWLEDGMENTS

This work was supported in part by the NSERC of Canada and the GACR (Grant No. 202/00/1026), GAUK (Grant No. 166/2000), UK (Grant No. 190-01/206053) and FRVS (Grant No. 1923) of the Czech Republic.

[1] A. Joseph, Rev. Mod. Phys. **39**, 829 (1967); C. A. Coulson and A. Joseph, *ibid.* **39**, 838 (1967).
[2] C. M. Bender and T. T. Wu, Phys. Rev. D **7**, 1620 (1973); T. I. Banks and C. M. Bender, J. Math. Phys. **13**, 1320 (1972); C. M. Bender and T. T. Wu, Phys. Rev. Lett. **27**, 461 (1971).
[3] J. Zinn-Justin, J. Math. Phys. **22**, 511 (1981).
[4] E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D **15**, 1544 (1977).
[5] J. C. Collins and D. E. Soper, Ann. Phys. (N.Y.) **112**, 209 (1978); G. Auberson, G. Mennessier and G. Mahoux, Nuovo Cimento A **46**, 1 (1978).
[6] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Nonrelativistic Theory* (Pergamon Press, Oxford, 1965).
[7] H. J. Silverstone, E. Harrell, and C. Grot, Phys. Rev. A **24**, 1925 (1981). Also see T. Yamabe, A. Tachibana, and H. J. Silverstone, *ibid.* **16**, 877 (1977); H. J. Silverstone, Int. J. Quantum Chem. **21**, 125 (1982); H. J. Silverstone, B. G. Adams, J. Čížek, and P. Otto, Phys. Rev. Lett. **43**, 1498 (1979); H. J. Silverstone, *ibid.* **55**, 2523 (1985).
[8] R. J. Damburg and V. V. Kolosov, J. Phys. B **11**, 1921 (1978). Also see R. J. Damburg and R. Kh. Propin, *ibid.* **16**, 3741 (1983).
[9] E. Harrell and B. Simon, Duke Math. J. **47**, 845 (1980). Also see E. M. Harrell, Int. J. Quantum Chem. **21**, 199 (1982).
[10] H. J. Silverstone, J. G. Harris, J. Čížek, and J. Paldus, Phys. Rev. A **32**, 1965 (1985).
[11] E. R. Vrscay, Phys. Rev. A **31**, 2054 (1985); Phys. Rev. Lett. **53**, 2521 (1984).
[12] J. S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967).
[13] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientist and Engineers* (McGraw-Hill, New York, 1978).
[14] W. Janke and H. Kleinert, Phys. Rev. A **42**, 2792 (1990).
[15] *Large-Order Behaviour of Perturbation Theory*, edited by J.

C. Le Guillou and J. Zinn-Justin (North-Holland, Amsterdam, 1990).

[16] J. Zamastil, J. Čížek, and L. Skála, Phys. Rev. Lett. **84**, 5683 (2000).

[17] *Proceedings of the Sanibel Workshop on Perturbation Theory at Large Order, Sanibel Conference, Florida, 1981*, edited by P. O. Löwdin and Y. Öhrn [Int. J. Quantum Chem. **21**, 1 (1982)].

[18] M. Reed and B. Simon, *Methods of Mathematical Physics IV: Analysis of Operators* (Academic, New York, 1978).

[19] B. Simon, Ann. Phys. (N.Y.) **58**, 76 (1970). Also see B. Simon, in *Cargese Lectures in Physics 1970*, edited by D. Bessis (Gordon and Breach, New York, 1972), Vol. 5, p. 383; C. M. Bender and T. T. Wu, Phys. Rev. **184**, 1231 (1969).

[20] E. J. Weniger, J. Čížek, and F. Vinette, J. Math. Phys. **34**, 571 (1993).

[21] J. Čížek and J. Paldus, Int. J. Quantum Chem. **12**, 875 (1977). Also see B. G. Adams, J. Čížek, and J. Paldus, Adv. Quantum Chem. **19**, 1 (1988). Reprinted in *Dynamical Groups and Spectrum Generating Algebras*, edited by A. O. Barut, A. Böhm, and Y. Neeman (World Scientific, Singapore, 1988), pp. 1–85; J. Čížek and F. Vinette, Theor. Chim. Acta **72**, 497 (1987); B. G. Adams, *Algebraic Approach to Simple Quantum Systems with Application to Perturbation Theory* (Springer-Verlag, Berlin, 1994).

[22] J. Čížek and E. R. Vrscay, Int. J. Quantum Chem. **21**, 27 (1982).

[23] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1965).

[24] C. M. Bender and G. V. Dunne, J. Math. Phys. **40**, 4616 (1999); R. Yaris, J. Bendler, R. A. Lovett, C. M. Bender, and P. A. Fetters, Phys. Rev. A **18**, 1816 (1978).