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From probabilities to quantum and classical mechanics

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Abstract

Probabilistic description of results of measurements and its consequences for understanding quantum mechanics are discussed. It is shown that basic mathematical structures of quantum mechanics like the probability amplitudes, Born rule, commutation and uncertainty relations, momentum operator, probability density current, rules for including the scalar and vector potentials and antiparticles can be obtained from the definition of the mean values of powers of the space coordinates and time. Equations of motion of quantum mechanics, the Klein–Gordon equation, Schrödinger equation and Dirac equation, are obtained from the requirement of relativistic invariance of the theory. Limit case of localized probability densities yields the Hamilton–Jacobi equation of classical mechanics. Many particle systems are also discussed.

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1. Introduction

Quantum mechanics is one of the most completely tested physical theories (see e.g. Refs. [1–3]). At the same time, the standard approach to introducing quantum mechanics based on the sometimes contra-intuitive postulates has a rather mathematical than physical character and the exact physical meaning of the postulates and their interpretation is the subject of continuing research. In this approach, quantum mechanics appears as a field with peculiar paradoxes and phenomena that are not easy to understand (see e.g. Ref. [4]). It is not satisfactory and, in our opinion, it is time (almost 80 years after formulating the basic principles of quantum mechanics) to replace this approach by a more physical one based on a more direct description of measurements. Only such an approach can clarify the physical meaning of

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assumptions made in quantum mechanics. Paraphrasing the title of the recent paper by Fuchs and Peres [5] we can say: Quantum theory needs no 'interpretation'—it needs derivation from description of measurements. In this sense, our approach can be understood as an extension and justification of standard interpretation of quantum mechanics.

We note that our approach is different from that used usually in physics: to explain the experimental results, one introduces some physical quantities and evolution equations for these quantities. Then, consequences of these equations are investigated and compared with experiment. In our approach, we describe the results of measurements in a probabilistic manner and ask what is the mathematical apparatus that can describe this situation. In this way, the basic mathematical structure of quantum mechanics except for equations of motion is obtained. Equations of motion are found from the requirement of the relativistic invariance of the probabilistic description.

In this paper, we do not discuss measurement processes in detail and assume that measuring apparatuses for measuring the space coordinates and time exist. A model based on the probabilistic description of the measured system and the measuring apparatus interacting with the thermodynamic bath can be found in Refs. [6,7]. In this sense, papers [6,7] and this work are complementary.

Probably the best approach is to start with measurement of the space coordinates and time. It is shown in this paper that the basic mathematical structure of quantum mechanics like the probability amplitudes, Born rule, commutation and uncertainty relations, momentum operator, probability density current, rules for including the scalar and vector potentials and antiparticles can be derived from the definition of the mean values of powers of the space coordinates and time (Sections 2–10). Equations of motion of quantum mechanics, the Klein–Gordon equation, Schrödinger equation and Dirac equation are obtained from the requirement of relativistic invariance of the theory (Sections 11–13). The limit case of localized probability densities yields the Hamilton–Jacobi equation of classical mechanics (Section 14). Many particle systems are discussed in Section 15.

2. Born rule

Physical measurements are imperfect and repeated measurements of the same quantity under the same experimental conditions yield different results. The most simple characteristic of such measurements is the mean value of results of repeated measurements. It is the starting point of the following discussion.

Results of repeated measurements of the coordinate x can be characterized by the mean values of x^n

$$\langle x^n \rangle = \int x^n \rho(\mathbf{r})\, \mathrm{d}V, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where integration is carried out over the whole space, $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and $\rho(\mathbf{r}) \geqslant 0$ is a normalized probability density

$$\int \rho\, \mathrm{d}V = \langle x^0 \rangle = 1. \tag{2}$$

First, we perform integration by parts with respect to the variable x in Eq. (1) and obtain

$$\frac{x^{n+1}}{n+1}\rho\bigg|_{x=-\infty}^{\infty} - \int \frac{x^{n+1}}{n+1}\frac{\partial \rho}{\partial x}\, \mathrm{d}V = \langle x^n \rangle. \tag{3}$$

Assuming that the first term in this equation equals zero for physically reasonable ρ we obtain

$$\int x^{n+1}\frac{\partial \rho}{\partial x}\, \mathrm{d}V = -(n+1)\langle x^n \rangle, \quad n = 0, 1, 2, \ldots . \tag{4}$$

We will show that this simple equation has remarkable consequences.

This equation can be rewritten in form of the inner product

$$(u, v) = -(n+1)\langle x^n \rangle \tag{5}$$

defined in the usual way

$$(u, v) = \int u^*(\mathbf{r}) v(\mathbf{r})\, \mathrm{d}V. \tag{6}$$

Here, the star denotes the complex conjugate and functions u and v can be taken in the general form

$$u = x^{n+1}\psi, \tag{7}$$

$$v = \frac{1}{\psi^*}\frac{\partial \rho}{\partial x}, \tag{8}$$

where $\psi = \psi(\mathbf{r})$ is an arbitrary complex function that will be discussed below. Due to Eq. (5), the Schwarz inequality $(u,u)(v,v) \geqslant |(u,v)|^2$ can be written in the form

$$(u,u)(v,v) \geqslant (n+1)^2 \langle x^n \rangle^2, \tag{9}$$

where

$$(u,u) = \int x^{2(n+1)} |\psi|^2 \, \mathrm{d}V \tag{10}$$

and

$$(v,v) = \int \frac{1}{|\psi|^2} \left(\frac{\partial \rho}{\partial x} \right)^2 \mathrm{d}V. \tag{11}$$

Till now, ψ could have been an arbitrary complex function and integrals (u,u) and (v,v) had no relation to $\langle x^n \rangle$. Now, we require that inequality (9) does not contain some abstract quantities (u,u) but physically relevant mean values of x^n and suppose

$$\int x^n \rho \, \mathrm{d}V = \int x^n |\psi|^2 \, \mathrm{d}V, \quad n = 2,4,6,\ldots. \tag{12}$$

These conditions do not determine the relation between ρ and $|\psi|^2$ uniquely. For this reason, we also assume that ρ and $|\psi|^2$ have the same norm

$$\int \rho \, \mathrm{d}V = \int |\psi|^2 \, \mathrm{d}V = 1 \tag{13}$$

and the same mean values of odd powers of x

$$\int x^n \rho \, \mathrm{d}V = \int x^n |\psi|^2 \, \mathrm{d}V, \quad n = 1,3,5,\ldots. \tag{14}$$

Then, we obtain the relation between the probability density ρ and probability amplitude ψ in the form of the Born rule

$$\rho = |\psi|^2 \tag{15}$$

or

$$\psi = \sqrt{\rho}\, \mathrm{e}^{\mathrm{i}s_1}, \tag{16}$$

where $s_1(\mathbf{r})$ is an arbitrary real function. The physical meaning of the phase s_1 will be discussed later.

We note that for $\rho = |\psi|^2$ the integral (v,v) is the so-called Fisher information [8–11].

The Born rule $\rho = |\psi|^2$ is the only relation between ρ and ψ for which $\int x^n \rho \, \mathrm{d}V = \int x^n |\psi|^2 \, \mathrm{d}V$, $n = 0,1,2,\ldots$. The significance of the Born rule follows from the fact that inequalities (9) contain for $\rho = |\psi|^2$ physically relevant quantities $\int x^n \rho \, \mathrm{d}V$, $n = 2,4,6,\ldots$. For other relations between ρ and ψ, this physically important property is in general lost.

3. Probability density current

We note that to describe physical systems, we have to specify not only the space–time probability distribution ρ but also its evolution in space–time. This information can be encoded into the complex part of the probability amplitude ψ. Here, we can proceed similarly as in continuum mechanics, where not only the density ρ but also the density current

$$j_k = \rho v_k, \quad k = 1,2,3 \tag{17}$$

related to components of the velocity v_k is introduced. Writing the "velocity" in the form

$$v_k = \frac{\partial s_1}{\partial x^k}, \tag{18}$$

where $s_1 = s_1(\mathbf{r}, t)$ is a real function we obtain

$$\begin{aligned} j_k &= \rho \frac{\partial s_1}{\partial x^k} = \sqrt{\rho}\mathrm{e}^{-\mathrm{i}s_1} \sqrt{\rho}\mathrm{e}^{\mathrm{i}s_1} \frac{\partial s_1}{\partial x^k} \\ &= \sqrt{\rho}\mathrm{e}^{-\mathrm{i}s_1}(-i) \frac{\partial(\sqrt{\rho}\mathrm{e}^{\mathrm{i}s_1})}{\partial x^k} + \frac{i}{2}\frac{\partial \rho}{\partial x^k}. \end{aligned} \tag{19}$$

Using the complex probability amplitude (16) we have

$$j_k = \psi^* \left(-i \frac{\partial \psi}{\partial x^k} \right) + \frac{i}{2} \frac{\partial \rho}{\partial x^k}. \tag{20}$$

However, the probability density current has to be real. Calculating the real part of j_k we obtain the final expression

$$\begin{aligned} j_k &= \frac{1}{2} \left[\psi^* \left(-i \frac{\partial \psi}{\partial x^k} \right) + \text{c.c.} \right] \\ &= \frac{1}{2i} \left(\psi^* \frac{\partial \psi}{\partial x^k} - \psi \frac{\partial \psi^*}{\partial x^k} \right). \end{aligned} \tag{21}$$

Except for a multiplicative factor, this formula agrees with the expression for the probability density current known from quantum mechanics. Complex probability amplitudes ψ are necessary to obtain nonzero j_k. The probability density current depends on the operator $-i(\partial/\partial x^k)$. Except for the factor $\hbar$, this operator agrees with the momentum operator $-i\hbar(\partial/\partial x^k)$ known from quantum mechanics. In agreement with rules of quantum mechanics, the probability amplitudes ψ and $\psi\exp(i\alpha)$, where α is a real constant, yield the same probability density ρ and probability density current j_k.

4. Commutation relation

Now we return back to Eq. (4) for $n=0$. Using the Born rule (15) we rewrite Eq. (4) in the form containing the probability amplitude ψ

$$\int x\left(\frac{\partial\psi^*}{\partial x}\psi+\psi^*\frac{\partial\psi}{\partial x}\right)\mathrm{d}V=-1. \tag{22}$$

Multiplying this equation by $-i$ we obtain the equation

$$\int\left[(x\psi)^*\left(-i\frac{\partial\psi}{\partial x}\right)-\left(-i\frac{\partial\psi}{\partial x}\right)^*x\psi\right]\mathrm{d}V=i \tag{23}$$

or in the operator form

$$[x,-i(\partial/\partial x)]=i. \tag{24}$$

This commutation relation is a straightforward consequence of Eq. (4) for $n=0$ and the Born rule (15) and need not be postulated. Except for the factor $\hbar$ determining the choice of units, this commutation relation agrees with the commutation relation between the coordinate and momentum operator known from quantum mechanics.

5. Uncertainty relations

The uncertainty relation for the coordinate x and the operator $-i(\partial/\partial x)$ can be derived in a standard way from the commutation relation (24) or, equivalently, by means of a simple calculation of (v,v)

$$\begin{aligned}(v,v)=&\int\frac{1}{|\psi|^2}\left(\frac{\partial\rho}{\partial x}\right)^2\mathrm{d}V\\ &=4\int\frac{1}{|\psi|^2}\left[\mathrm{Re}\left(\psi^*\frac{\partial\psi}{\partial x}\right)\right]^2\mathrm{d}V\\ &\leqslant 4\int\left|-i\frac{\partial\psi}{\partial x}\right|^2\mathrm{d}V.\end{aligned} \tag{25}$$

Substituting expressions (10) and (25) into inequality (9) for $n=0$ we obtain, except for $\hbar^2$, the uncertainty relation in the form known from quantum mechanics

$$\int x^2|\psi|^2\,\mathrm{d}V\int\left|-i\frac{\partial\psi}{\partial x}\right|^2\mathrm{d}V\geqslant\frac{1}{4}. \tag{26}$$

This result can be further generalized. Using integration by parts and the condition $\rho\to 0$ for $x\to\pm\infty$ Eq. (22) can be generalized as

$$\begin{aligned}&\int[(x-a)\psi]^*\left[\frac{\partial\psi}{\partial x}-ib\psi\right]\mathrm{d}V\\ &\quad+\int\left[\frac{\partial\psi}{\partial x}-ib\psi\right]^*[(x-a)\psi]\,\mathrm{d}V=-1,\end{aligned} \tag{27}$$

where a and b are real constants. From this equation, a more general form of the uncertainty relation can be obtained

$$\int(x-a)^2|\psi|^2\,\mathrm{d}V\int|-i(\partial\psi/\partial x)-b\psi|^2\,\mathrm{d}V\geqslant\frac{1}{4}. \tag{28}$$

The minimum of the left-hand side is obtained for

$$a=\int\psi^* x\psi\,\mathrm{d}V=\langle x\rangle \tag{29}$$

and

$$b=\int\psi^*(-i\partial\psi/\partial x)\,\mathrm{d}V=\langle -i\partial/\partial x\rangle. \tag{30}$$

Except for the factor $\hbar$, the resulting uncertainty relation with a and b given by the last two equations agrees with the well-known Heisenberg uncertainty relation. Again, it can be obtained from Eq. (4) for $n=0$ and the Born rule (15).

Uncertainty relations are a general consequence of Eq. (1) and must appear in any probabilistic theory of this kind, including quantum mechanics. There are two important quantities appearing in the uncertainty relations: the coordinate x and the operator $-i(\partial/\partial x)$. Similar quantities, namely the coordinate x and the momentum operator $-i\hbar(\partial/\partial x)$, also appear in quantum mechanics.

6. Vector potential

It is worth noting that Eq. (27) also remains valid in the case where b is a real function $b = f_x(\mathbf{r}, t)$. It means that the operator $-i(\partial\psi/\partial x)$ can be replaced by the operator $-i(\partial\psi/\partial x) - f_x$ and the commutation relation (24) and the uncertainty relation (28) can be generalized further. Therefore, the general structure of the probability theory remains preserved for any real function f_x. In physics, functions f_x, f_y and f_z can for example correspond to different components of the electromagnetic vector potential $\mathbf{A} = (A_x, A_y, A_z)$ multiplied by the charge q of the particle. Except for $\hbar$ and q, it agrees with the rule $-i\hbar\nabla \to -i\hbar\nabla - q\mathbf{A}$ for including the vector potential $\mathbf{A}$ into quantum theory. We note also that the kinetic energy in quantum mechanics $T = (\hbar^2/2m)\int|\nabla\psi|^2\,\mathrm{d}V$ equals for real ψ the space Fisher information $\int|\nabla\rho|^2/\rho\,\mathrm{d}V$ multiplied by $\hbar^2/(8m)$.

7. Time

Time can be discussed similarly as the space coordinates; however, there are some important differences that have to be taken into consideration.

Assuming that there are given initial conditions for $\psi(\mathbf{r}, t = 0)$ the probability amplitude $\psi(\mathbf{r}, t)$, $t > 0$ gives a probabilistic description of the results of measurements at later times. Therefore, time evolution has a unidirectional character from the given initial conditions to the relative probability of results of (yet unperformed) measurements at later times. If this measurement is actually performed, probabilistic description must be replaced by a concrete result following from the performed measurement. It is the basis of two different evolution schemes in quantum mechanics: time evolution described by the evolution equation like the Schrödinger equation and reduction or collapse of the wave function. In this paper, we are interested in the former case. Discussion of the latter case can be found for example in Refs. [6,7].

In standard quantum mechanics, the probability amplitudes obey the normalization condition $\int|\psi|^2\,\mathrm{d}V = 1$ valid at all times and integral $\int_0^\infty\int|\psi|^2\,\mathrm{d}V\,\mathrm{d}t$ goes to infinity. This situation can be compared to that for a free particle. For a free particle, integral $\int|\psi|^2\,\mathrm{d}V$ goes to infinity and ψ is usually normalized by means of the Dirac δ-function. For time, a similar approach cannot be used for two reasons. First, here we do not perform integration over all times, but from the initial condition at $t = 0$ to infinity. Second, if the integral $\int_0^\infty\int|\psi|^2\,\mathrm{d}V\,\mathrm{d}t$ goes to infinity we cannot define the mean time by analogy with Eq. (1) and proceed similarly as in the preceding sections. For these reasons, we assume that not only the integral $\int|\psi|^2\,\mathrm{d}V$ but also the integral

$$\int_0^\infty\int|\psi|^2\,\mathrm{d}V\,\mathrm{d}t = 1 \tag{31}$$

equals one and proceed by analogy with the space coordinates. In this way, we obtain the operator $i(\partial/\partial t)$, obtain the corresponding commutation and uncertainty relations and introduce the scalar potential. At the end of our discussion, we will assume that $\int|\psi|^2\,\mathrm{d}V$ changes negligibly in time, normalize the probability amplitude by means of the usual condition $\int|\psi|^2\,\mathrm{d}V = 1$ and perform transition to standard quantum mechanics.

8. Time component of the probability density current

First, we define the time component of the probability density current by the equation analogous to Eqs. (17)–(18)

$$j_t = -\rho\,\frac{\partial s}{\partial t} \tag{32}$$

and obtain an expression similar to Eq. (21)

$$j_t = \frac{1}{2}\left[\psi^*\left(i\,\frac{\partial\psi}{\partial t}\right) + \text{c.c.}\right]. \tag{33}$$

Except for a factor, this quantity equals the time component of the probability density current $j_0 = \mathrm{Re}[\psi^* i\hbar(\partial\psi/\partial x^0)]/m_0$ known from relativistic quantum mechanics, where $x^0 = ct$ and m_0 is the rest mass. Then, by analogy with Eq. (23) we derive the equation

$$\int_{t=0}^{\infty}\int\left[\left(i\,\frac{\partial\psi}{\partial t}\right)^{*}t\psi-(t\psi)^{*}\left(i\,\frac{\partial\psi}{\partial t}\right)\right]\mathrm{d}V\,\mathrm{d}t=i. \tag{34}$$

One can also introduce a real constant d into this equation

$$\int_{t=0}^{\infty}\int\left[\left(i\,\frac{\partial\psi}{\partial t}-d\psi\right)^{*}t\psi -(t\psi)^{*}\left(i\,\frac{\partial\psi}{\partial t}-d\psi\right)\right]\mathrm{d}V\,\mathrm{d}t=i. \tag{35}$$

The uncertainty relation for time can be written in a form analogous to Eq. (28)

$$\int_{0}^{\infty}\int t^{2}|\psi|^{2}\,\mathrm{d}V\,\mathrm{d}t\int_{0}^{\infty}\int\left|i(\partial\psi/\partial t)-d\psi\right|^{2}\mathrm{d}V\,\mathrm{d}t \geqslant\frac{1}{4}. \tag{36}$$

Minimum of the left-hand side is obtained for

$$d=\frac{1}{2}\left[\int_{0}^{\infty}\int\psi^{*}i(\partial\psi/\partial t)\,\mathrm{d}V\,\mathrm{d}t+\text{c.c.}\right]. \tag{37}$$

Eq. (36) is also valid if d is replaced by a real function $f_0(\mathbf{r},t)$.

To illustrate the physical meaning of Eq. (36) we assume a decaying probability amplitude with the lifetime $\tau>0$

$$\psi(\mathbf{r},t)=\frac{1}{\sqrt{\tau}}\,\mathrm{e}^{-\mathrm{i}(\omega t-t)/(2\tau)}\psi(\mathbf{r}), \tag{38}$$

where the space part of the probability amplitude is normalized by the usual condition $\int|\psi(\mathbf{r})|^2\,\mathrm{d}V=1$. In this case, from Eqs. (36)–(37) we obtain

$$\int_{0}^{\infty}\int t^{2}|\psi|^{2}\,\mathrm{d}V\,\mathrm{d}t=2\tau^{2}, \tag{39}$$

$d=\omega$ and

$$\int_{0}^{\infty}\int|i(\partial\psi/\partial t)-d\psi|^{2}\,\mathrm{d}V\,\mathrm{d}t=\frac{1}{4\tau^{2}}. \tag{40}$$

Therefore, uncertainty relation (36) is the relation between the mean square time $\langle t^2\rangle=2\tau^2$ and the square of the imaginary part of the complex frequency $\omega-i/(2\tau)$ and has the meaning of the time–energy uncertainty relation.

9. Antiparticles

In agreement with our understanding of direction of time, we assume that direct physical meanings have only the probability amplitudes corresponding to the non-negative values of the time component of the probability density current integrated over the whole space

$$\int j_t\,\mathrm{d}V=-\int\rho\,\frac{\partial s}{\partial t}\,\mathrm{d}V\geqslant 0. \tag{41}$$

If this quantity is negative, its sign can be reversed by the transformation $\psi\to\psi^*$ changing the sign of the phase s and the probability density currents j_k and j_t. Performing this transformation from Eq. (35) for $d=f_0$, where f_0 is a real function, we obtain

$$\int_{t=0}^{\infty}\int\left[\left(i\,\frac{\partial\psi}{\partial t}+f_0\psi\right)^{*}t\psi -(t\psi)^{*}\left(i\,\frac{\partial\psi}{\partial t}+f_0\psi\right)\right]\mathrm{d}V\,\mathrm{d}t=i \tag{42}$$

and see that this transformation changes the sign of f_0.

A similar discussion can also be carried out for the space coordinates. As a result, transformation $\psi\to\psi^*$ leads to a change of sign of the functions f_0 and f_k, $k=1,2,3$ that can be respected by putting $f_0=qU$ and $f_k=qA_k$, where U and A_k are the scalar and vector electromagnetic potentials. Therefore, the probability amplitudes ψ and ψ^* describe particles that differ by the sign of their charge and the existence of particles and antiparticles agrees with the general structure of the probability theory and unidirectional character of time.

Except for $\hbar$, these results agree with the rules $i\hbar(\partial/\partial t)\to i\hbar(\partial/\partial t)-qU$ and $-i\hbar\nabla\to -i\hbar\nabla-q\mathbf{A}$ for including the electromagnetic potentials in quantum mechanics. These potentials representing different physical scenarios do not appear among the variables of the probability amplitude and describe non-quantized classical fields.

10. Standard quantum mechanics

Now, we perform a transition to standard quantum mechanics. In this limit case the integral

$\int |\psi|^2 \, dV$ does not depend on time and the probability density can be normalized by the usual condition $\int |\psi|^2 \, dV = 1$. At the same time, uncertainty relation (36) loses its original meaning and time becomes a parameter rather than a dynamical variable. This is the first reason for a different role of time and space coordinates in quantum mechanics. The second reason is that the operator $i(\partial/\partial t)$ appears in equations of motion like the Schrödinger equation and does not represent an independent physical quantity.

It is worth noting that to obtain results of Sections 2–9 no evolution equation was required. Therefore, this part of the mathematical formalism of quantum mechanics follows directly from the probabilistic description of results of measurements. It is also interesting that the Planck constant $\hbar$ does not appear in our discussion and can be included by multiplying Eqs. (23) and (34) by $\hbar$. Therefore, the Planck constant determines units used in measurements and scales at which the probabilistic character of measurements is important.

11. Equations of motion

To find equations of motion we require relativistic invariance of the theory. Our approach is similar to that used by Frieden who derived basic equations of physics from the principle of extreme physical information [10].

First we note that all quantities discussed above depend on ψ or its first derivatives with respect to the space coordinates and time. Returning back to our scheme used in Sections 7–9 we can use the first derivatives of ψ and create the relativistic invariant

$$\int_0^\infty \int \left(\frac{1}{c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 - \sum_{k=1}^{3} \left| \frac{\partial \psi}{\partial x^k} \right|^2 \right) dV \, dt = \text{const}, \tag{43}$$

where c is the speed of light and *const* is a constant independent of t, x^k and ψ. Integral $\int_0^\infty \int |\partial\psi/\partial t|^2 \, dV \, dt$ has the meaning of the time Fisher information and is non-negative. A similar conclusion also applies for $\int_0^\infty \int |\partial\psi/\partial x^k|^2 \, dV \, dt$, $k = 1, 2, 3$. However, since Eq. (43) must be valid for all possible ψ including the case $\partial\psi/\partial x^k = 0$ (in the language of quantum mechanics, it corresponds to zero momentum and zero kinetic energy) we can conclude that const$\geqslant 0$.

12. Klein–Gordon equation

In Eq. (43), we can perform integration by parts with respect to all variables. For example, we obtain for time

$$\int_0^\infty \int \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} \, dV \, dt = \frac{1}{2} \left[\int \left(\psi^* \frac{\partial \psi}{\partial t} + \text{c.c.} \right) dV \right]_0^\infty - \frac{1}{2} \int_0^\infty \int \left(\psi^* \frac{\partial^2 \psi}{\partial t^2} + \text{c.c.} \right) dV \, dt. \tag{44}$$

However, the first integral on the right-hand side can be expressed as $\partial(\int |\psi|^2 \, dV)/\partial t$ and disappears in the limit of standard quantum mechanics when $\int |\psi|^2 \, dV = 1$. An analogous result can also be obtained for the space variables x^k assuming that $\partial|\psi|^2/\partial x^k$ for $x^k \to -\infty$ and $x^k \to \infty$ equal. In standard quantum mechanics, this condition is obeyed for a free particle as well as for the bound states.

Now, we perform the transition to standard quantum mechanics with the probability amplitude normalized in the usual way and obtain an equation that has to be valid for all ψ

$$\frac{1}{2} \int \left[\psi^* \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \text{const} \right) \psi + \text{c.c.} \right] dV = 0. \tag{45}$$

This condition is obeyed if the probability amplitudes ψ fulfil the equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \text{const} \right) \psi = 0. \tag{46}$$

Since const$\geqslant 0$, we can put const $= m_0^2 c^2/\hbar^2$, where m_0 is another constant, known as the rest mass of the particle. Therefore, the requirement of relativistic invariance applied to quantities appearing in the probabilistic formulation leads to the Klein–Gordon equation for a free particle.

The non-relativistic time Schrödinger equation can be obtained from the Klein–Gordon equation by using the transformation

$$\psi = \mathrm{e}^{m_0 c^2 t/(i\hbar)} \varphi, \tag{47}$$

where φ is the probability amplitude appearing in the Schrödinger equation. This transition is known and will not be discussed here [12].

13. Dirac equation

The Dirac equation can be obtained by replacing the probability amplitude ψ in Eq. (43) by a column vector with four components

$$\int_0^\infty \int \left(\frac{1}{c^2} \frac{\partial \psi^+}{\partial t} \frac{\partial \psi}{\partial t} - \sum_{k=1}^{3} \frac{\partial \psi^+}{\partial x^k} \frac{\partial \psi}{\partial x^k} \right) \mathrm{d}V \, \mathrm{d}t = \text{const}, \tag{48}$$

where the cross denotes the Hermitian conjugate. Inserting the γ^μ matrices [12] into this equation, putting $\text{const} = m_0^2 c^2/\hbar^2$ and using Eq. (31) we obtain

$$\int_0^\infty \int \left[\frac{1}{c^2} \left(\gamma^0 \frac{\partial \psi}{\partial t} \right)^+ \left(\gamma^0 \frac{\partial \psi}{\partial t} \right) - \sum_{k=1}^{3} \left(\gamma^k \frac{\partial \psi}{\partial x^k} \right)^+ \left(\gamma^k \frac{\partial \psi}{\partial x^k} \right) - \frac{m_0^2 c^2}{\hbar^2} \psi^+ \psi \right] \mathrm{d}V \, \mathrm{d}t = 0. \tag{49}$$

Then, using properties of the γ^μ matrices and assuming that integration by parts can be used analogously as in the case of Eq. (44) the last equation leads in the limit of standard quantum mechanics to (see also Refs. [9,10])

$$\int \left(\frac{\gamma^0}{c} \frac{\partial \psi}{\partial t} - \sum_{k=1}^{3} \gamma^k \frac{\partial \psi}{\partial x^k} - \frac{i m_0 c}{\hbar} \psi \right)^+ \times \left(\frac{\gamma^0}{c} \frac{\partial \psi}{\partial t} + \sum_{k=1}^{3} \gamma^k \frac{\partial \psi}{\partial x^k} + \frac{i m_0 c}{\hbar} \psi \right) \mathrm{d}V = 0. \tag{50}$$

The operator in the first parentheses is the Hermitian conjugate of that in the second ones. Assuming that the expression in the second parentheses equals zero we obtain the Dirac equation for a free particle

$$\frac{\gamma^0}{c} \frac{\partial \psi}{\partial t} + \sum_{k=1}^{3} \gamma^k \frac{\partial \psi}{\partial x^k} + \frac{i m_0 c}{\hbar} \psi = 0. \tag{51}$$

We see that the requirement of relativistic invariance of the probabilistic description yields all basic equations of motion of quantum mechanics. The scalar and vector potentials can be included by means of the rules $i\hbar(\partial/\partial t) \to i\hbar(\partial/\partial t) - qU$ and $-i\hbar\nabla \to -i\hbar\nabla - q\mathbf{A}$ discussed above.

14. Classical mechanics

To derive the Hamilton–Jacobi equation for a free particle we proceed as follows: the probability amplitude is assumed in the form

$$\psi = \mathrm{e}^{\mathrm{i}s/\hbar} = \mathrm{e}^{\mathrm{i}s_1/\hbar} \mathrm{e}^{-s_2/\hbar}, \tag{52}$$

where s_1 and s_2 are the real and imaginary parts of s, respectively. In the limit of standard quantum mechanics mentioned above, Eq. (43) with $\text{const} = m_0^2 c^2/\hbar^2$ can be replaced by the equation

$$\frac{1}{c^2} \int \left| \frac{\partial s}{\partial t} \right|^2 |\psi|^2 \, \mathrm{d}V = \int |\nabla s|^2 |\psi|^2 \, \mathrm{d}V + m_0^2 c^2. \tag{53}$$

Now we assume that the probability density

$$\rho = |\psi|^2 = \mathrm{e}^{-2s_2/\hbar} \tag{54}$$

has very small values everywhere except in the vicinity of the point $\langle \mathbf{r} \rangle$, where it achieves its maximum and the first derivatives of s_2 at this point equal zero

$$\left. \frac{\partial s_2}{\partial x^k} \right|_{\mathbf{r}=\langle \mathbf{r} \rangle} = 0, \quad k = 1, 2, 3. \tag{55}$$

In such a case, the probability density can be replaced by the δ-function

$$|\psi|^2 = \delta(\mathbf{r} - \langle \mathbf{r} \rangle) \tag{56}$$

and the probabilistic character of the theory disappears. Eqs. (53)–(56) then lead to the relativistic equation

$$\frac{1}{c^2} \left(\frac{\partial s_1(\langle \mathbf{r} \rangle, \langle t \rangle)}{\partial t} \right)^2 = [\nabla s_1(\langle \mathbf{r} \rangle, t)]^2 + m_0^2 c^2. \tag{57}$$

We note that Eq. (56) corresponds to the limit $\hbar \to 0$ in Eq. (54). Therefore, s_1 in Eq. (57) is in fact the first term of the expansion of s_1 into the power series in $\hbar$

$$s_1 = s_1|_{\hbar=0}. \tag{58}$$

Further, we replace the mean coordinates $\langle \mathbf{r} \rangle$ by $\mathbf{r}$ as is usual in classical mechanics and introduce the classical non-relativistic action $S(\mathbf{r}, t)$

$$s_1 = S - m_0 c^2 t. \tag{59}$$

Eq. (57) then leads to

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - m_0 c^2\right)^2 = (\nabla S)^2 + m_0^2 c^2. \tag{60}$$

In the non-relativistic limit $|\partial S/\partial t| \ll m_0 c^2$ the last equation yields the Hamilton–Jacobi equation for a free particle

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m_0} = 0. \tag{61}$$

Thus, the Hamilton–Jacobi equation can be obtained from the probabilistic description of measurements in the limit of δ-like probability densities and non-relativistic approximation. The scalar and vector potentials U and $\mathbf{A}$ can be included by means of the rules $\partial S/\partial t \to \partial S/\partial t + qU$ and $\nabla S \to \nabla S - q\mathbf{A}$ following from the rules $i\hbar(\partial/\partial t) \to i\hbar(\partial/\partial t) - qU$ and $-i\hbar\nabla \to -i\hbar\nabla - q\mathbf{A}$ discussed above.

15. Many-particle systems

In general, many particle systems have to be described by quantum field theory. However, if we limit ourselves to quantum mechanics, we can proceed as follows:

The starting point of discussion of the N particle system is a definition analogous to Eq. (1)

$$\langle \mathbf{r}_j \rangle = \int \mathbf{r}_j \rho(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)\, \mathrm{d}V_1 \ldots \mathrm{d}V_N, \quad j = 1, \ldots, N, \tag{62}$$

where ρ is the many-particle probability density and $\mathbf{r}_j$ are the coordinates of the jth particle. Then, discussion can be carried out analogous to that given above and the probability amplitude, uncertainty and commutation relations, momentum operators and density currents for all particles can be introduced. The scalar and vector potentials $U(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)$ and $\mathbf{A}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)$ and antiparticles can also be discussed.

Equations of motion for N free particles can be found from generalization of the relativistic invariant (43)

$$\int_0^\infty \int \left(\frac{1}{c^2}\left|\frac{\partial \psi}{\partial t}\right|^2 - \sum_{j=1}^{N} |\nabla_j \psi|^2\right) \mathrm{d}V_1 \ldots \mathrm{d}V_N\, \mathrm{d}t = \sum_{j=1}^{N} \frac{m_j^2 c^2}{\hbar^2}, \tag{63}$$

where $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)$ is the N particle probability amplitude and m_j denotes the rest mass of the particle.

Using a similar approach as above, we can then obtain the N particle Schrödinger equation

$$-\frac{\hbar^2}{2m_j} \sum_{j=1}^{N} \Delta_j \psi = i\hbar \frac{\partial \psi}{\partial t} \tag{64}$$

and the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{N} \frac{(\nabla_j S)^2}{2m_j} = 0. \tag{65}$$

For a system of identical particles, the probability amplitude must be symmetric or antisymmetric with respect to exchange of any two particles i and j.

The non-locality of quantum mechanics is related to the many-particle character of the probability density ρ and the corresponding probability amplitude ψ.

It is seen that the probabilistic description of results of measurements and its relativistic invariance also yields the basic mathematical structure of many-particle quantum mechanics.

16. Conclusions

In this paper, we have shown that the basic mathematical structure of quantum mechanics can be obtained from the definition of the mean values of the powers of the space coordinates and time. Equations of motion of quantum mechanics have been obtained from the requirement of relativistic invariance of the theory. As a limit case, this approach also yields the Hamilton–Jacobi equation of classical mechanics.

Since our approach makes it possible to obtain the most significant parts of mathematical formalism of quantum mechanics from the probabilistic description of results of measurements, we believe that it is a natural and physically satisfactory starting point for understanding this field. It also contributes to understanding quantum theory as a correctly formulated probabilistic description of measurements that can describe physical phenomena at different levels of accuracy from the most simple models to very complex ones.

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