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Renormalized Perturbation Theory for Quartic Anharmonic Oscillator

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The analytic structure of the renormalized energy of the quartic anharmonic oscillator described by the Hamiltonian $H = p^2 + x^2 + \beta x^4$ is discussed and the dispersion relation for the renormalized energy is found. It follows from the analytic structure that the renormalized strong coupling expansion converges not only for all positive values of the coupling constant β but also for some double-well problems. Further, exact dispersion relations for the weak and strong coupling expansion coefficients of the renormalized energy are derived. The large-order formulas for these coefficients found in previous papers follow simply from the dispersion relations. The renormalized weak coupling expansion is separated into the Stieltjes and non-Stieltjes parts. Numerical tests performed for the ground and first excited states confirm correctness of our conclusions. Finally, properties of different perturbative approaches to the anharmonic oscillator are compared.

I. INTRODUCTION

In this paper, we investigate the Schrödinger equation for the anharmonic oscillator

$$H\psi = E\psi, \tag{1}$$

$$H = p^2 + x^2 + \beta x^4, \qquad \beta > 0, \tag{2}$$

where $p = -id/dx$. As is wellknown, the weak coupling expansion for the energy $E = E(\beta)$,

$$E(\beta) = \sum_{n=0}^{\infty} b_n \beta^n, \tag{3}$$

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diverges for an arbitrary value of the coupling constant $\beta \in (0, \infty)$ [1–3]. The energy $E(\beta)$ has also the strong coupling expansion

$$E(\beta) = \beta^{1/3} \sum_{n=0}^{\infty} K_n \beta^{-2n/3}, \tag{4}$$

which converges for sufficiently large β [3].

The renormalization approach has many advantages [4–8]. It leads to the Schrödinger equation

$$H_R \psi = E_R(\kappa) \psi, \tag{5}$$

$$H_R = p^2 + x^2 + \kappa(x^4/3 - x^2), \tag{6}$$

where

$$E_R(\kappa) = (1-\kappa)^{1/2} E(\beta) \tag{7}$$

and

$$\beta = \frac{\kappa}{3(1-\kappa)^{3/2}}. \tag{8}$$

Similarly to the ordinary energy $E(\beta)$, the renormalized energy $E_R(\kappa)$ has a weak coupling expansion

$$E_R(\kappa) = \sum_{n=0}^{\infty} c_n \kappa^n \tag{9}$$

divergent for all $\kappa \in (0, 1]$ [5, 9] and a strong coupling expansion

$$E_R(\kappa) = \sum_{n=0}^{\infty} \Gamma_n (1-\kappa)^n, \tag{10}$$

which has the most favorable properties from all the expansions mentioned above. The results of [7, 8] show that, in contrast to the expansions (3), (4), and (9), the expansion (10) converges for all $\kappa \in (0, 1]$, i.e., for all $\beta > 0$.

The transformation described by Eqs. (7) and (8) is a special case of a more general transformation discussed in [10], where the convergence of the delta expansion and order dependent mappings (DE–ODM) for the quartic oscillator was proven. It was shown in [10] that, except for a numerical factor, the transformation (7)–(8) is the only transformation between $E(\beta)$ and $E_R(\kappa)$ for which the convergence of the DE–ODM can be proven. We note that instead of this rather mathematical approach other, physically motivated arguments were used in [4] to introduce the renormalization (7)–(8).

We note also that the renormalization introduced in [4] is closely related to the Wick ordering of the Hamiltonian H [2, 11],

$$:H: = p^2 + x^2 + \beta x^4 - 3\beta x^2 + 3\beta/4 - 1. \tag{11}$$

The large-order behavior of the weak coupling expansion coefficients of the energy corresponding to the Hamiltonian (11) was investigated in [2, 11]. However, the dispersion relation for these coefficients given in [11] is valid for large n only. The exact dispersion relation for the weak coupling coefficients c_n is discussed in this paper.

The main purpose of this paper is to prove that the renormalized strong coupling expansion (10) is the *convergent* Taylor series in the circle $|1-\kappa|<1$, i.e., the expansion (10) converges for all $\kappa\in(0, 2)$. To achieve this aim we first clarify the analytic structure of the renormalized energy $E_R(\kappa)$. From this analytic structure, exact dispersion relations for the energy $E_R(\kappa)$ and c_n and Γ_n coefficients are found. It is shown that the large-order formulas for the c_n and Γ_n coefficients found in previous papers [2, 5, 7, 8, 11] follow simply from these dispersion relations. The summation rules for the Γ_n coefficients are also discussed.

Formulation of the convergent perturbation theory is important not only from the point of view of the one-dimensional quartic oscillator but also from the point of view of more complex multidimensional problems in which the standard weak coupling perturbation approaches lead to divergent asymptotic series (for example, the Stark and Zeemann effect for the hydrogen atom or the vibrational problem of molecules [12]). Discussion of the convergent perturbation theory for the quartic oscillator is only a first step in this program.

The paper is organized as follows. Following [2], we first summarize in Section II the derivation of the dispersion relation for the ordinary energy $E(\beta)$ and exact and large-order formulas for the b_n coefficients. In Section IIIA, the analytic structure of the renormalized energy $E_R(\kappa)$ is investigated. Our discussion of the analytic structure of the energy $E_R(\kappa)$ is similar to that given in [10] in the context of the DE–ODM method. However, instead of investigating the remainder of the weak coupling series with order dependent transformation (8), we keep the transformation fixed and find the dispersion relation for the renormalized energy $E_R(\kappa)$ (Section IIIB). It appears that the energy $E_R(\kappa)$ can be written as a sum of the Stieltjes and non-Stieltjes parts. Using the dispersion relation, we find in Sections IV and V dispersion relations for the c_n and Γ_n coefficients. The large-order formulas for these coefficients follow simply from the dispersion relations. It is shown in Section V that, in contrast to the divergent weak coupling expansion (9), the strong coupling expansion (10) converges for all $\kappa\in(0, 2)$. In Section VI, summation rules for the Γ_n coefficients are discussed. Numerical results supporting our theoretical discussion are presented in Section VII. In Appendixes A and B, calculation of integrals needed in Sections IV, V, and VII is described.

II. ORDINARY ENERGY $E(\beta)$ AND b_n COEFFICIENTS

In this section, we summarize analytic properties of the ordinary energy $E(\beta)$, derivation of the dispersion relation for the expansion coefficients b_n, and the large-order formula for the b_n coefficients.

The dispersion relation for the ordinary energy $E(\beta)$ follows from the rigorous results for $E(\beta)$ as a function of a complex variable β [1, 3, 13].

The energy $E(\beta)$ given by Eq. (1) has the following structure on a three-sheeted Riemann surface $\arg\beta \in (-3\pi, 3\pi)$. On the principal branch $\arg\beta \in (-\pi, \pi)$, the energy $E(\beta)$ is analytic and has a cut at $\arg\beta = \pi$. On the remaining sheets, the energy $E(\beta)$ has sequences of square-root branch points with the asymptotic phase $\arg\beta = 3\pi/2$ and the limit point of the branch points at the origin. Therefore, the singular point $\beta = 0$ is not an isolated singularity. Other sequences of the branch points are given by the property

$$E(\beta) = -E(e^{i3\pi}\beta). \tag{12}$$

For the ground state energy, sequences of the branch points begin at the radius $\beta_{\min} = 0.09746483$ [14] corresponding to the radius of convergence of the series (4). In the case of the first excited state energy, sequences of the branch points begin at $\beta_{\min} = 0.06165$ [15]. For the second and third excited states, sequences of the branch points begin at the same radius $\beta_{\min}$ as that for the ground and first excited states, respectively. For higher excited states, values of $\beta_{\min}$ go down [14, 15]. For $|\beta| > \beta_{\min}$, the energy $E(\beta)$ is analytic on the three-sheeted Riemann surface for all $\arg\beta \in (-3\pi, 3\pi)$.

Thus, the expansion (4) converges for all complex β for which $|\beta| > \beta_{\min}$, $\arg\beta \in (-3\pi, 3\pi)$.

At the singular point $\beta = 0$, Eq. (1) becomes the Schrödinger equation of the harmonic oscillator with the energy

$$E(0) = 2K + 1, \tag{13}$$

where $K = 0, 1, 2, \ldots$ is the index of the excitation.

It is seen from Eq. (4) that for $\beta \to \infty$ the energy $E(\beta)$ behaves as

$$E(\beta \to \infty) \to \beta^{1/3} K_0. \tag{14}$$

To derive the dispersion relation for the energy $E(\beta)$ we can use the Cauchy theorem. Our approach is only slightly different from that used in [2]. Because of the analyticity of the energy $E(\beta)$ on the principal branch $\arg\beta \in (-\pi, \pi)$ we can write

$$E(\beta) = E(0) + \frac{1}{2\pi i} \oint_C d\tilde{\beta}\, \frac{E(\tilde{\beta}) - E(0)}{\tilde{\beta} - \beta}, \qquad \beta > 0, \tag{15}$$

where the integration path C is shown in Fig. 1. For the reason which is clarified below, the constant term $E(0)$ is written separately. For $\beta = 0$ the quartic oscillator becomes the harmonic oscillator. For this reason, Eq. (15) can be used also in the limit $\beta \to 0+$.

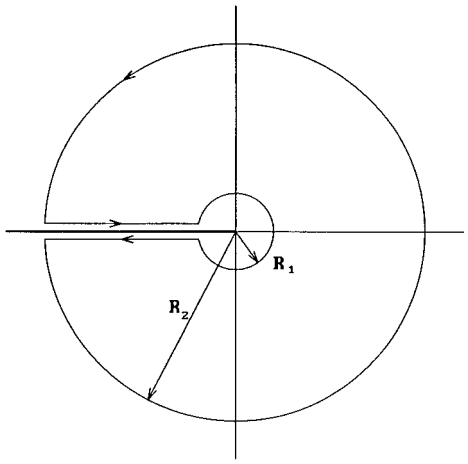

FIG. 1. Integration path in the complex β or κ plane.

The last equation can be written in the form

$$E(\beta)=E(0)+\frac{1}{2\pi i}\int_{-R_2}^{-R_1}d\tilde{\beta}\,\frac{\Delta E(\tilde{\beta})}{\tilde{\beta}-\beta}+\frac{1}{2\pi}\int_{\pi}^{-\pi}d\varphi\,[E(R_1e^{i\varphi})-E(0)]\,\frac{R_1e^{i\varphi}}{R_1e^{i\varphi}-\beta}+\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\,[E(R_2e^{i\varphi})-E(0)]\,\frac{R_2e^{i\varphi}}{R_2e^{i\varphi}-\beta},\tag{16}$$

where

$$\Delta E(\tilde{\beta})=E(\tilde{\beta}+i\varepsilon)-E(\tilde{\beta}-i\varepsilon),\qquad \varepsilon\to 0+\tag{17}$$

is the discontinuity of $E(\beta)$ along the negative real axis. It is obvious that the integral along the circle with the radius R_1 vanishes for $R_1\to 0$. Expanding the fractions in the remaining two integrals in Eq. (16) into the geometric series in $\beta/\tilde{\beta}$ and $\beta/[R_2e^{i\varphi}]$ we get

$$E(\beta)=E(0)+\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{-R_2}^{0}d\tilde{\beta}\,\frac{\Delta E(\tilde{\beta})}{\tilde{\beta}^{n+1}}+\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\,\frac{E(R_2e^{i\varphi})-E(0)}{(R_2e^{i\varphi})^n}\right]\beta^n.\tag{18}$$

Here, we took into account the fact that because of the equation

$$\lim_{\beta\to 0+}\oint_C d\tilde{\beta}\,\frac{E(\tilde{\beta})-E(0)}{\tilde{\beta}-\beta}=\int_{-R_2}^{0}d\tilde{\beta}\,\frac{\Delta E(\tilde{\beta})}{\tilde{\beta}}+i\int_{-\pi}^{\pi}d\varphi\,[E(R_2e^{i\varphi})-E(0)]=0\tag{19}$$

following from Eq. (15) these expansions do not contain $n=0$ terms. Since $|E(\beta)| \sim |\beta|^{1/3}$ for $\beta \to \infty$, the integrals in Eq. (18) along the circle with the radius R_2 vanish for $R_2 \to \infty$.

It follows from Eqs. (3), (13), (18), and (19) that

$$b_0 = 2K + 1. \tag{20}$$

The remaining b_n coefficients can be calculated from the exact dispersion relation [2, 3] following from Eqs. (3) and (18) for $R_2 \to \infty$,

$$b_n = \frac{1}{2\pi i} \int_{-\infty}^{0} d\tilde{\beta}\, \frac{\Delta E(\tilde{\beta})}{\tilde{\beta}^{n+1}}, \qquad n \geqslant 1. \tag{21}$$

It is seen from Eq. (4) that the discontinuity of the ordinary energy $\Delta E(\beta)$ along the negative real axis

$$\Delta E(\beta) = \sum_{n=0}^{\infty} K_n [(|\beta|\, e^{i\pi})^{(1-2n)/3} - (|\beta|\, e^{-i\pi})^{(1-2n)/3}] \tag{22}$$

is given by the discontinuity of the function $\beta^{(1-2n)/3}$. Here, we take $\arg \beta \in (0, \pi)$ in the upper half plane and $\arg \beta \in (-\pi, 0)$ in the lower half plane. Equation (22) can be used in the region of convergence of the series (4).

To find the large-order behavior of the b_n coefficients it is sufficient to calculate the discontinuity of the energy $\Delta E(\beta)$ for small $\beta < 0$. For this aim, we cannot use the expansion (22), which diverges for small β. Since $E(\beta)$ is the eigenvalue of the hermitian operator, the discontinuity of the energy $\Delta E(\beta)$ is purely imaginary. For $\beta < 0$, the Hamiltonian H given by Eq. (2) has no bound states and the reciprocal value of $|\Delta E(\beta)|$ represents the lifetime of the quasistationary states. For small $\beta < 0$ it is possible to calculate this lifetime by means of the JWKB method. The discontinuity of the energy $\Delta E(\beta)$ in the lowest order of JWKB equals [2, 3, 16]

$$\Delta E(\beta) = i\, \frac{2^{K+2}}{\pi^{1/2} K!} \left(-\frac{4}{\beta}\right)^{K+1/2} \exp\left(\frac{2}{3\beta}\right). \tag{23}$$

The large-order formula for the b_n coefficients can be obtained directly from Eqs. (21) and (23) [2]

$$b_n = (-1)^{n+1}\, \frac{2\sqrt{6}}{\pi^{3/2}}\, \frac{12^K}{K!} \left(\frac{3}{2}\right)^n \Gamma(n + K + 1/2), \qquad n \gg 1. \tag{24}$$

It is seen from this equation that the series (3) diverges for any $\beta > 0$.

III. RENORMALIZED ENERGY E_R

To clarify properties of the weak coupling and strong coupling expansions (9) and (10) of the renormalized energy $E_R(\kappa)$ we proceed analogously to the previous section. To calculate the perturbation coefficients c_n and Γ_n we use the Cauchy theorem and derive the corresponding dispersion relations. For this aim, we have to know the analytic structure of the renormalized energy $E_R(\kappa)$ given by Eq. (7). We investigate its properties on the sheet $\arg \kappa \in (-\pi, \pi)$.

A. *Analytic Structure of the Renormalized Energy*

For $0 < |\kappa| < 1$, Eq. (8) transforms the sheet $\arg \kappa \in (-\pi, \pi)$ into the sheet $\arg \beta \in (-\pi, \pi)$. Since the energy $E(\beta = \kappa/[3(1-\kappa)^{3/2}])$ and the function $(1-\kappa)^{1/2}$ are in this region analytic, the renormalized energy $E_R(\kappa)$ is for $0 < |\kappa| < 1$, $\arg \kappa \in (-\pi, \pi)$ also analytic.

For $|\kappa| \geqslant 1$, Eq. (8) transforms the region $\arg \kappa \in (-\pi, \pi)$ into the region $\arg \beta \in (-3\pi/2, 3\pi/2)$. As mentioned in Section II, the energy $E(\beta)$ is for $|\beta| > \beta_{\min}$ analytic on the three-sheeted Riemann surface. However, the factor $(1-\kappa)^{1/2}$ in Eq. (7) and the transformation (8) have discontinuity along the real axis for $\kappa \geqslant 1$. Taking $1-\kappa = |1-\kappa| \exp(i\pi)$ if we approach the real axis from the upper half plane and $1-\kappa = |1-\kappa| \exp(-i\pi)$ if we approach the real axis from the lower half plane we get

$$E_R(|1-\kappa|\, e^{\pm i\pi}) = |(1-\kappa)^{1/2}|\, e^{\pm i\pi/2}\, E(|\beta|\, e^{\mp i3\pi/2}), \tag{25}$$

where

$$|\beta| = \frac{|\kappa|}{3\,|1-\kappa|^{3/2}}. \tag{26}$$

However, it follows from Eq. (12) that the discontinuity of the energy $E_R(\kappa)$ for real $\kappa \geqslant 1$ equals zero

$$\begin{aligned} &E_R(|1-\kappa|\, e^{i\pi}) - E_R(|1-\kappa|\, e^{-i\pi}) \\ &\quad = |(1-\kappa)^{1/2}|\, e^{i\pi/2}[E(|\beta|\, e^{-i3\pi/2}) + E(|\beta|\, e^{i3\pi/2})] = 0. \end{aligned} \tag{27}$$

Therefore, the energy $E_R(\kappa)$ is on the sheet $\arg \kappa \in (-\pi, \pi)$ analytic at least for $\kappa_{\min} \leqslant |\kappa| < \kappa_{\max}$, where $\kappa_{\min}$ and $\kappa_{\max}$ correspond to $\beta_{\min}$ via Eq. (8). The values of $\kappa_{\min}$ and $\kappa_{\max}$ can be determined as follows. The function $\beta = \beta(\kappa)$ given by Eq. (8) has a minimum for $\arg \kappa = \arg \beta = \pi$. The resulting equation

$$|\beta| = |\kappa|/[3(1+|\kappa|)^{3/2}] \tag{28}$$

has for given $\beta_{\min}$ two positive real roots $\kappa_{\min}$ and $\kappa_{\max}$. For the ground state, we get $\kappa_{\min} = 0.5815$ and $\kappa_{\max} = 8.321$. For the first excited state, we have $\kappa_{\min} = 0.262$ and $\kappa_{\max} = 26.1$. As mentioned in Section II, the values of $\beta_{\min}$ for higher excited states go down. It means that the corresponding $\kappa_{\min}$ go down and $\kappa_{\max}$ go up.

Thus, the renormalized energy E_R can be on the sheet $\arg\kappa\in(-\pi,\pi)$ and $\kappa_{\min}<|\kappa|<\kappa_{\max}$ expressed by the convergent series

$$E_R(\kappa)=\left(\frac{\kappa}{3}\right)^{1/3}\sum_{n=0}^{\infty}K_n\left[\left(\frac{\kappa}{3}\right)^{-2/3}(1-\kappa)\right]^n, \tag{29}$$

following from Eqs. (4), (7), and (8).

For $|\kappa|>\kappa_{\max}$ and $|\arg(1-\kappa)|>2\pi/3$ the branch points of $E(\beta)$ are transformed by Eq. (7) from the sheets $\arg\beta\in(-3\pi/2, 3\pi/2)$ to the sheet $\arg\kappa\in(-\pi,\pi)$. Therefore, $E_R(\kappa)$ is not analytic in this region.

Now we discuss the point $\kappa=0$ corresponding to $\beta=0$. It follows from Eqs. (7) and (8) that, similarly to the ordinary energy $E(\beta)$, the renormalized energy $E_R(\kappa)$ is singular at $\kappa=0$. For $\kappa=0$, the Hamiltonian (6) becomes the Hamiltonian of the harmonic oscillator. Therefore, we get

$$E_R(\kappa=0)=E(\beta=0)=2K+1. \tag{30}$$

The point $\kappa=1$ corresponds to $\beta\to\infty$. It follows from Eqs. (4) and (7) that for $\beta\to\infty$

$$E_R(\kappa)=(1-\kappa)^{1/2}\,\beta^{1/3}K_0. \tag{31}$$

It is seen from this equation and Eqs. (8) and (10) that

$$E_R(\kappa=1)=\Gamma_0=K_0/3^{1/3}. \tag{32}$$

It means that, in contrast to the ordinary energy, the renormalized energy remains finite at the point $\kappa=1$.

Our discussion can be summarized as follows. *The renormalized energy* $E_R(\kappa)$ *given by Eq.* (7) *is analytic in the sheet* $\arg\kappa\in(-\pi,\pi)$ *at least for* $0<|\kappa|<\kappa_{\max}$. Outside of this region non-analytic behavior related to the branch points of the energy $E(\beta)$ can be expected. Since $\kappa_{\max}>2$, we get as a special case that *the renormalized energy* $E_R(\kappa)$ *is in the sheet* $\arg\kappa\in(-\pi,\pi)$ *analytic everywhere in the circle* $|\kappa-1|<1$. These results are used in the following sections.

We note that these conclusions about the analytic structure of the renormalized energy agree with those obtained within the DE–ODM method [10].

B. *Dispersion Relation for the Renormalized Energy*

Now it is obvious that we can use the Cauchy theorem for the energy $E_R(\kappa)$ with the integration path shown in Fig. 1, where $R_2<\kappa_{\max}$. In contrast to the ordinary energy $E(\beta)$, we cannot perform the limit $R_2\to\infty$ here. We get

$$E_R(\kappa)=\frac{1}{2\pi i}\int_{-R_2}^{-R_1}d\tilde{\kappa}\,\frac{\Delta E_R(\tilde{\kappa})}{\tilde{\kappa}-\kappa}+\frac{1}{2\pi}\int_{\pi}^{-\pi}d\varphi\,E_R(R_1e^{i\varphi})\,\frac{R_1e^{i\varphi}}{R_1e^{i\varphi}-\kappa}$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\,E_R(R_2e^{i\varphi})\,\frac{R_2e^{i\varphi}}{R_2e^{i\varphi}-\kappa}, \tag{33}$$

where $R_2 < \kappa_{\max}$. For $\kappa = 0$ the quartic oscillator becomes the harmonic oscillator. For this reason, Eq. (33) can be used also in the limit $\kappa \to 0+$. Assuming $R_1 \to 0$, the energy $E_R(\kappa)$ behaves as $E_R(\kappa) \to 2K+1$ and the second integral vanishes. To calculate $E_R(\kappa)$ from Eq. (33) it is sufficient to know the discontinuity $\Delta E_R(\kappa)$ along the negative real axis and to calculate the integral along the circle with the radius R_2. The final formula for $E_R(\kappa)$ reads

$$E_R(\kappa) = E_R^{(1)}(\kappa) + E_R^{(2)}(\kappa), \tag{34}$$

where

$$E_R^{(1)}(\kappa) = \frac{1}{2\pi i} \int_{-R_2}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{\tilde{\kappa} - \kappa} \tag{35}$$

is the contribution from the negative real axis and

$$E_R^{(2)}(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, E_R(R_2 e^{i\varphi})\, \frac{R_2 e^{i\varphi}}{R_2 e^{i\varphi} - \kappa} \tag{36}$$

is the contribution from the circle with the radius R_2.

It follows from Eq. (7) that the discontinuity of the renormalized energy along the negative real axis equals

$$\Delta E_R(\kappa) = (1-\kappa)^{1/2}\, \Delta E(\beta). \tag{37}$$

In the lowest order of κ we get from Eqs. (23) and (37)

$$\Delta E_R(\kappa) = i\, \frac{2^{K+2}}{\pi^{1/2} e^3 K!} \left(-\frac{12}{\kappa}\right)^{K+1/2} \exp\left(\frac{2}{\kappa}\right). \tag{38}$$

This equation represents the lowest order of the JWKB approximation valid for small $\kappa < 0$. Using the method suggested in [16] we derived Eq. (38) also directly by means of the JWKB method applied to Eq. (5) for small $\kappa < 0$. Equation (38) will be used for the derivation of the large-order formulas for the coefficients c_n and Γ_n in Sections IV and V.

At the end of this section, we show that $-E_R^{(1)}(\kappa)$ is the Stieltjes function.

It can be proven directly from the definition of the Stieltjes function (see, for example, [3, 17, 18]). Substituting $\tilde{\kappa} = -1/x$ into Eq. (35) we get

$$-E_R^{(1)}(\kappa) = \int_0^{\infty} \frac{d\psi_R(x)}{1+\kappa x}, \tag{39}$$

where

$$d\psi_R(x) = 1/(2\pi i)\, \Delta E_R(-1/x)(1/x)\, dx, \qquad x \geqslant 1/R_2 \tag{40}$$

and

$$d\psi_R(x) = 0, \qquad x < 1/R_2. \tag{41}$$

Now we use the fact that the function

$$E(0) - E(\beta) = \frac{1}{2\pi i} \int_{-\infty}^{0} d\tilde{\beta}\, \frac{\Delta E(\tilde{\beta})}{\beta - \tilde{\beta}} = \int_0^{\infty} \frac{d\psi(z)}{1+\beta z}, \tag{42}$$

where $\tilde{\beta} = -1/z$ and

$$d\psi(z) = 1/(2\pi i)\, \Delta E(-1/z)(1/z)\, dz \tag{43}$$

is the Stieltjes function in the extended sense [3]. Therefore, the measure $d\psi(z)$ is nonnegative. Using Eq. (37) it is seen that also the measure $d\psi_R(x) = 0$ for $x \in (0, 1/R_2)$ and $d\psi_R(x) = (1+1/x)^{1/2}\, (z/x)(d\psi(z)/dz)\, dx$ for $x \in (1/R_2, \infty)$ is nonnegative for all $x \in (0, \infty)$. Therefore, $-E_R^{(1)}(\kappa)$ is the Stieltjes function. This result is used for a numerical test of the correctness of the dispersion relation (34)–(36) in Section VII.

We note that the approximate measures $d\psi(z)$ and $d\psi_R(x)$ given by Eqs. (23), (43) and (38), (40)–(41) obey the conditions $d\psi(z) \geqslant 0$ and $d\psi_R(x) \geqslant 0$.

IV. COEFFICIENTS c_n

In this section, we discuss the exact dispersion relation for the weak coupling expansion coefficients c_n and derive the large-order formula for these coefficients. The reason for the divergence of the weak coupling series (9) is clarified.

The dispersion relation for the c_n coefficients can be derived analogously to Eq. (21) for the b_n coefficients. Equations (35) and (36) can be written in the form

$$E_R^{(1)}(\kappa) = \frac{1}{2\pi i} \int_{-R_2}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{\tilde{\kappa}}\, \frac{1}{1 - \dfrac{\kappa}{\tilde{\kappa}}} \tag{44}$$

and

$$E_R^{(2)}(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, E_R(R_2 e^{i\varphi})\, \frac{1}{1 - \dfrac{\kappa}{R_2 e^{i\varphi}}}. \tag{45}$$

Expanding the last fractions in these equations into the geometric series we get the exact dispersion relation for the c_n coefficients

$$c_n = c_n^{(1)} + c_n^{(2)}, \qquad n \geqslant 0, \tag{46}$$

where the $c_n^{(1)}$ coefficient is the contribution from the negative real axis

$$c_n^{(1)} = \frac{1}{2\pi i} \int_{-R_2}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{\tilde{\kappa}^{n+1}} \tag{47}$$

and $\Delta E_R(\tilde{\kappa})$ is given by Eq. (37). The second term, $c_n^{(2)}$, is the contribution from the circle with the radius R_2

$$c_n^{(2)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, \frac{E_R(R_2 e^{i\varphi})}{(R_2 e^{i\varphi})^n}, \tag{48}$$

where $E_R(\kappa)$ is given by Eq. (29). The integral in the last equation can be calculated analytically (see Appendix A).

Now we show that the contribution of the $c_n^{(2)}$ coefficients to c_n can be neglected for large n.

First we find the large-order formula for the $c_n^{(1)}$ coefficients. For large n, the dominant contribution to the integral (47) is given by $\tilde{\kappa} \to 0-$. Therefore, we can replace R_2 in Eq. (47) by infinity and get

$$c_n^{(1)} = \frac{1}{2\pi i} \int_{-\infty}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{\tilde{\kappa}^{n+1}}, \qquad n \gg 1. \tag{49}$$

Substituting Eq. (38) into Eq. (49) we obtain the leading term of the large-order formula for the coefficients $c_n^{(1)}$

$$c_n^{(1)} = (-1)^{n+1} \frac{2\sqrt{6}}{\pi^{3/2} e^3} \frac{12^K}{K!} \frac{1}{2^n} \Gamma(n + K + 1/2), \qquad n \gg 1. \tag{50}$$

Now we discuss the $c_n^{(2)}$ contribution to c_n. Taking the absolute value of $c_n^{(2)}$ and denoting

$$\max_{\varphi \in (-\pi, \pi)} |E_R(R_2 e^{i\varphi})| = C \tag{51}$$

we get the estimate

$$|c_n^{(2)}| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, \frac{E_R(R_2 e^{i\varphi})}{(R_2 e^{i\varphi})^n} \right| \leqslant \frac{C}{R_2^n}. \tag{52}$$

It is obvious that for $1 < R_2 < \kappa_{\max}$ the $c_n^{(2)}$ contribution to the c_n coefficients can be neglected for large n. Thus, the large-order behavior of the c_n coefficients is given by the formula

$$c_n = (-1)^{n+1} \frac{2\sqrt{6}}{\pi^{3/2} e^3} \frac{12^K}{K!} \frac{1}{2^n} \Gamma(n + K + 1/2), \qquad n \gg 1 \tag{53}$$

in agreement with [2, 5, 11]. It follows from Eqs. (24) and (53) that the ratio of the b_n and c_n coefficients equals

$$\frac{b_n}{c_n} = e^3 3^n, \qquad n \gg 1. \tag{54}$$

We see that the weak coupling expansion of the renormalized ground state energy $E_R(\kappa)$ diverges for any $\kappa > 0$ similarly to the weak coupling expansion of the ordinary energy $E(\beta)$. In both cases, the divergent parts of the coefficients c_n and b_n are given by the integration of the discontinuity of the energies $\Delta E_R(\kappa)$ and $\Delta E(\beta)$ in the vicinity of the expansion points $\kappa = 0$ and $\beta = 0$. We see in agreement with [9, 11] that the divergence of the weak coupling expansions is a general feature of the anharmonic oscillators.

We note that the dispersion relation for the c_n coefficients in the form of Eq. (49) was given already in [11]. We have shown here that it is the approximate relation which can be used for large n only. The exact dispersion relation is given by Eqs. (46)–(48).

V. COEFFICIENTS Γ_n

In this section we show that the strong coupling expansion (10) is a convergent series. We find the exact dispersion relation for the Γ_n coefficients and derive the large-order formula for these coefficients. This large-order formula provides more information on the convergence of the expansion.

The renormalized strong coupling expansion (10) is the expansion of $E_R(\kappa)$ at the point $\kappa = 1$. It follows from Section III that the energy $E_R(\kappa)$ is analytic in the circle $|\kappa - 1| < 1$ with the center at the point $\kappa = 1$ corresponding to $\beta = \infty$. Thus, *the series* (10) *is the convergent Taylor expansion in this circle and converges for all* $\kappa \in (0, 2)$. This result extends the proof of [6], which was given for the neighborhood of $\kappa = 1$ only.

The exact dispersion formula for the Γ_n coefficients can be obtained analogously to that for the c_n coefficients. If we use the integration path along the circle with the center at $\kappa = 1$ and radius R_2 (see Fig. 2), we can write equations for $E_R^{(1)}(\kappa)$ and $E_R^{(2)}(\kappa)$ analogous to Eqs. (35) and (36)

$$E_R^{(1)}(\kappa) = \frac{1}{2\pi i} \int_{-R_2+1}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{\tilde{\kappa} - 1}\, \frac{1}{1 - \dfrac{\kappa - 1}{\tilde{\kappa} - 1}} \tag{55}$$

and

$$E_R^{(2)}(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, \frac{E_R(R_2 e^{i\varphi} + 1)}{1 - \dfrac{\kappa - 1}{R_2 e^{i\varphi}}}. \tag{56}$$

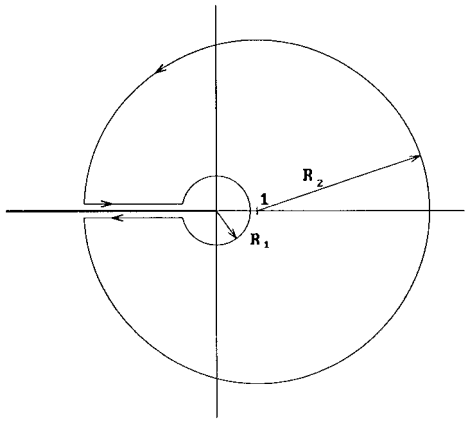

FIG. 2. Integration path in the complex κ plane.

Expanding the last fractions in these equations into the geometric series we get the exact dispersion relation for the Γ_n coefficients

$$\Gamma_n = \Gamma_n^{(1)} + \Gamma_n^{(2)}, \qquad n \geqslant 0, \tag{57}$$

where

$$\Gamma_n^{(1)} = \frac{(-1)^n}{2\pi i} \int_{-R_2+1}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{(\tilde{\kappa}-1)^{n+1}} \tag{58}$$

and $\Delta E_R(\tilde{\kappa})$ is given by Eq. (37). The second term equals

$$\Gamma_n^{(2)} = \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} d\varphi\, \frac{E_R(R_2 e^{i\varphi}+1)}{(R_2 e^{i\varphi})^n}, \tag{59}$$

where $E_R(\kappa)$ is given by Eq. (29). The integral in the last equation can be calculated analytically (see Appendix B).

Now we show that, analogously to the $c_n^{(2)}$ coefficients, the contribution of the $\Gamma_n^{(2)}$ coefficients to Γ_n can be neglected for large n.

For large n, the dominant contribution to the integral (58) is given by $\tilde{\kappa} \to 0-$. Therefore, we can replace R_2 in Eq. (58) by infinity and get

$$\Gamma_n^{(1)} = \frac{(-1)^n}{2\pi i} \int_{-\infty}^{0} d\tilde{\kappa}\, \frac{\Delta E_R(\tilde{\kappa})}{(\tilde{\kappa}-1)^{n+1}}, \qquad n \gg 1. \tag{60}$$

Inserting Eq. (38) into Eq. (60) we obtain after some manipulation the leading term of the large-order behavior of the coefficients $\Gamma_n^{(1)}$

$$\Gamma_n^{(1)} = -\frac{4\sqrt{6}}{\pi^{3/2} e^3}\, \frac{12^K}{K!} \int_0^{\infty} dt\, \frac{t^{K-3/2} e^{-t}}{(1+2/t)^{n+1}}. \tag{61}$$

Equation (61) is equivalent to Eq. (23) in [8] for $m = 2$.

To find the analytic large-order formula for the $\Gamma_n^{(1)}$ coefficients we rewrite Eq. (61) as

$$\Gamma_n^{(1)} = -\frac{4\sqrt{6}}{\pi^{3/2} e^3} \frac{12^K}{K!} \int_0^\infty dt\, t^{K-3/2}\, e^{-[t+(n+1)\ln(1+2/t)]}, \tag{62}$$

and use the Laplace method [17] to calculate the integral of the form

$$\int_a^b dx\, \varphi(x)\, e^{-f(x)} \approx \varphi(x_0) \left(\frac{2\pi}{f''(x)|_{x=x_0}} \right)^{1/2} e^{-f(x_0)}. \tag{63}$$

Here, x_0 is the point where $f'(x)|_{x=x_0} = 0$. The large-order formula for the $\Gamma_n^{(1)}$ coefficients then reads

$$\Gamma_n^{(1)} = -\frac{4\sqrt{6}}{\pi e^2} \frac{12^K}{K!} (2n)^{(K-1)/2}\, e^{-2\sqrt{2n}}. \tag{64}$$

Now we discuss the $\Gamma_n^{(2)}$ contribution to Γ_n. Taking the absolute value of $\Gamma_n^{(2)}$ and denoting

$$\max_{\varphi \in (-\pi, \pi)} |E_R(R_2 e^{i\varphi} + 1)| = D \tag{65}$$

we get the estimate

$$|\Gamma_n^{(2)}| = \left| \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} d\varphi\, \frac{E_R(R_2 e^{i\varphi} + 1)}{(R_2 e^{i\varphi})^n} \right| \leqslant \frac{D}{R_2^n}. \tag{66}$$

It is obvious that taking sufficiently large R_2 obeying the condition $1 < R_2 < \kappa_{\max} - 1$ we can neglect the $\Gamma_n^{(2)}$ contribution to the Γ_n coefficients for large n. Thus, the final large-order formula for the Γ_n coefficients is given by the $\Gamma_n^{(1)}$ part

$$\Gamma_n = -\frac{4\sqrt{6}}{\pi e^2} \frac{12^K}{K!} (2n)^{(K-1)/2}\, e^{-2\sqrt{2n}}. \tag{67}$$

This result was first obtained from the analysis of the numerically computed coefficients Γ_n [7]. Later, it was derived from the large-order formula for the c_n coefficients (53) [8]. Our approach represents direct and more rigorous derivation without using the large-order formula for the c_n coefficients.

Using, for example, the d'Alembert convergence criterion it follows from Eq. (67) that, in agreement with our conclusion made at the beginning of this section, the strong coupling expansion (10) converges for all $\kappa \in (0, 1]$ corresponding to $\beta \in (0, \infty)$. It is seen, however, that it converges also for $\kappa \in [1, 2)$ corresponding to the double-well problem with the Hamiltonian (6) written in the form

$$H_R = p^2 + \kappa x^4/3 + (1 - \kappa) x^2. \tag{68}$$

We note also that in the region of analyticity of the energy $E_R(\kappa)$ the series (10) can be analytically continued to $\kappa > 2$ [19].

VI. SUMMATION RULES

In this section, we derive summation rules describing the mutual relation of the $\Gamma_n^{(i)}$ and $c_n^{(i)}$ coefficients.

Using the weak and strong coupling expansions of the renormalized energy we get

$$E_R^{(i)} = \sum_{n=0}^{\infty} c_n^{(i)} \kappa^n = \sum_{n=0}^{\infty} \Gamma_n^{(i)} (1-\kappa)^n, \qquad i = 1, 2. \tag{69}$$

Putting $\kappa = 0$ we have the summation rule

$$c_0^{(i)} = \sum_{n=0}^{\infty} \Gamma_n^{(i)}, \qquad i = 1, 2. \tag{70}$$

Calculating the derivatives of the series in Eq. (69) at the point $\kappa = 0$ we get the general summation rule

$$(-1)^j j!\, c_j^{(i)} = \sum_{n=j}^{\infty} n(n-1)\cdots(n-j+1)\Gamma_n^{(i)}, \qquad i = 1, 2. \tag{71}$$

From Eqs. (70) and (71), we get also the summation rule for the c_n and Γ_n coefficients

$$(-1)^j j!\, c_j = \sum_{n=j}^{\infty} n(n-1)\cdots(n-j+1)\Gamma_n, \qquad j \geqslant 0, \tag{72}$$

which was derived in [7]. For $j = 0$ we get as a special case [6]

$$c_0 = 2K + 1 = \sum_{n=0}^{\infty} \Gamma_n. \tag{73}$$

These rules can be used for numerical tests of the computed coefficients.

It follows from Eq. (69) for $i = 1$ and Sections IV and V that the large-order formula for the Γ_n coefficients (67) can be obtained from the large-order formula for the c_n coefficients (53) as was done in [8].

VII. NUMERICAL RESULTS

To clarify the structure of the coefficients c_n and Γ_n, we first compute the numerical values of the $c_n^{(1)}$, $c_n^{(2)}$ and $\Gamma_n^{(1)}$, $\Gamma_n^{(2)}$ coefficients. Then, we verify the summation

rules given in Section VI and check the correctness of the dispersion relation (34)–(36).

To perform numerical tests, we used numerical values of 200 hundred Γ_n coefficients [7] for the ground and first excited states of the quartic oscillator calculated by the method described in [20].

From these coefficients, we calculated the K_n coefficients via the equations [21]

$$K_0 = 3^{1/3}\Gamma_0 \tag{74}$$

and

$$K_n 3^{(2n-1)/3} = \Gamma_n - \sum_{i=0}^{n-1} K_i \frac{3^{(2i-1)/3}}{(n-i)!} \frac{\Gamma\left(\frac{2i-1}{3} + n - i\right)}{\Gamma\left(\frac{2i-1}{3}\right)} \tag{75}$$

following from a comparison of the series (4) and (10).

The exact c_n coefficients in the form of rational numbers were calculated by the method described in [1, 2, 5].

Then, using Eqs. (48), (59), (29) and the integrals given in Appendixes A and B we calculated the $c_n^{(2)}$ and $\Gamma_n^{(2)}$ coefficients. To calculate these coefficients by the method described in Appendixes A and B we used 200 coefficients K_n for the ground state and 100 coefficients K_n for the first excited state. For the ground state, the summation of 200 terms of the series (4) gives good results for $|\beta| > 0.12$. To achieve high accuracy of the renormalized energy $E_R(\kappa)$ we must take $|\beta|$ as large as possible. It is obvious that the function (26), where $\kappa = |\kappa| \exp(i\varphi)$ and $|\kappa|$ is fixed, has a minimum at $\varphi = \pi$. Equation (8) transforms $\kappa \in (-\infty, 0)$ into $\beta \in (-\frac{2}{27}3^{1/2}, 0)$. This transformation yields the maximum absolute value of β for $\kappa = -2$. For this reason, we used $R_2 = 2$ in our calculations.

The remaining coefficients, $c_n^{(1)}$ and $\Gamma_n^{(1)}$, were found by means of Eqs. (46) and (57),

$$\begin{aligned} c_n^{(1)} &= c_n - c_n^{(2)}, \\ \Gamma_n^{(1)} &= \Gamma_n - \Gamma_n^{(2)}, \end{aligned} \tag{76}$$

where c_n and Γ_n are the exact values of the coefficients. We note that the $c_n^{(1)}$ and $\Gamma_n^{(1)}$ coefficients could be derived also directly from the discontinuity of the energy $E_R(\kappa)$ (see Eqs. (47) and (58)). However, bearing in mind the magnitude of the coefficients $c_n^{(2)}$ and $\Gamma_n^{(2)}$, this would require extremely precise knowledge of the discontinuity $\Delta E_R(\kappa)$. We shall discuss this point in a future paper.

The values of the $c_n^{(1)}$, $c_n^{(2)}$, $\Gamma_n^{(1)}$, and $\Gamma_n^{(2)}$, coefficients, $n = 0, ..., 20$, for the ground and first excited states are shown in Tables I and II. In contrast to the $c_n^{(i)}$ coefficients, the $\Gamma_n^{(i)}$ coefficients for $n > 3$ do not change sign. It is seen that while the absolute value of the $c_n^{(1)}$ coefficients increases rapidly with n, the absolute value of

TABLE I

The Stieltjes $c_n^{(1)}$ and Complementary Non-Stieltjes $c_n^{(2)}$ Parts of the Weak Coupling Coefficients $c_n = c_n^{(1)} + c_n^{(2)}$ for the Ground and First Excited States of the Quartic Oscillator

	Ground state		First excited state	
n	$c_n^{(1)}$	$c_n^{(2)}$	$c_n^{(1)}$	$c_n^{(2)}$
0	$-0.100\,069\,885\,466\,10^{-1}$	$0.101\,000\,698\,854\,10^{1}$	$-0.165\,635\,963\,434\,10^{0}$	$3.165\,635\,963\,434\,10^{0}$
1	$0.928\,242\,652\,198\,10^{-2}$	$-0.259\,282\,426\,521\,10^{0}$	$0.176\,176\,392\,948\,10^{0}$	$-0.426\,176\,392\,948\,10^{0}$
2	$-0.105\,810\,829\,154\,10^{-1}$	$-0.102\,522\,504\,178\,10^{-1}$	$-0.245\,748\,984\,665\,10^{0}$	$-0.250\,843\,486\,678\,10^{-1}$
3	$0.153\,462\,205\,783\,10^{-1}$	$0.278\,779\,421\,614\,10^{-3}$	$0.456\,131\,462\,468\,10^{0}$	$0.741\,020\,419\,771\,10^{-2}$
4	$-0.284\,179\,989\,398\,10^{-1}$	$-0.191\,665\,412\,043\,10^{-3}$	$-0.109\,007\,334\,488\,10^{1}$	$-0.277\,243\,058\,038\,10^{-2}$
5	$0.656\,845\,453\,406\,10^{-1}$	$0.797\,052\,380\,852\,10^{-4}$	$0.319\,481\,167\,843\,10^{1}$	$0.110\,665\,200\,392\,10^{-2}$
6	$-0.183\,664\,121\,343\,10^{0}$	$-0.329\,865\,446\,038\,10^{-4}$	$-0.110\,271\,140\,845\,10^{2}$	$-0.459\,614\,745\,328\,10^{-3}$
7	$0.604\,018\,229\,272\,10^{0}$	$0.141\,537\,706\,558\,10^{-4}$	$0.435\,807\,857\,850\,10^{2}$	$0.196\,344\,713\,689\,10^{-3}$
8	$-0.228\,519\,139\,018\,10^{1}$	$-0.619\,169\,620\,786\,10^{-5}$	$-0.193\,495\,963\,738\,10^{3}$	$-0.856\,516\,889\,809\,10^{-4}$
9	$0.977\,777\,391\,384\,10^{1}$	$0.274\,991\,830\,903\,10^{-5}$	$0.952\,281\,261\,653\,10^{3}$	$0.379\,705\,217\,858\,10^{-4}$
10	$-0.466\,877\,447\,273\,10^{2}$	$-0.123\,639\,076\,589\,10^{-6}$	$-0.514\,329\,842\,115\,10^{4}$	$-0.170\,488\,315\,814\,10^{-4}$
11	$0.246\,122\,512\,190\,10^{3}$	$0.561\,476\,902\,473\,10^{-5}$	$0.302\,488\,330\,486\,10^{5}$	$0.773\,447\,496\,455\,10^{-5}$
12	$-0.141\,989\,283\,124\,10^{4}$	$-0.257\,106\,865\,532\,10^{-6}$	$-0.192\,482\,954\,395\,10^{6}$	$-0.353\,896\,090\,138\,10^{-5}$
13	$0.889\,835\,365\,807\,10^{4}$	$0.118\,559\,612\,639\,10^{-6}$	$0.131\,809\,699\,864\,10^{7}$	$0.163\,092\,984\,344\,10^{-5}$
14	$-0.601\,990\,326\,326\,10^{5}$	$-0.549\,994\,095\,420\,10^{-7}$	$-0.966\,809\,952\,330\,10^{7}$	$-0.756\,217\,186\,624\,10^{-6}$
15	$0.437\,271\,926\,240\,10^{6}$	$0.256\,462\,339\,658\,10^{-7}$	$0.756\,450\,999\,423\,10^{8}$	$0.352\,486\,257\,768\,10^{-6}$
16	$-0.339\,436\,572\,920\,10^{7}$	$-0.120\,128\,676\,886\,10^{-7}$	$-0.629\,023\,404\,810\,10^{9}$	$-0.165\,053\,936\,704\,10^{-6}$
17	$0.280\,429\,410\,049\,10^{8}$	$0.564\,926\,247\,459\,10^{-8}$	$0.554\,064\,596\,102\,10^{10}$	$0.775\,988\,365\,675\,10^{-7}$
18	$-0.245\,678\,064\,717\,10^{9}$	$-0.266\,601\,771\,340\,10^{-8}$	$-0.515\,418\,988\,888\,10^{11}$	$-0.366\,124\,981\,484\,10^{-7}$
19	$0.227\,499\,489\,716\,10^{10}$	$0.126\,210\,692\,698\,10^{-8}$	$0.504\,992\,841\,239\,10^{12}$	$0.173\,292\,671\,035\,10^{-7}$
20	$-0.222\,026\,839\,099\,10^{11}$	$-0.599\,172\,377\,799\,10^{-9}$	$-0.519\,826\,182\,209\,10^{13}$	$-0.822\,556\,074\,122\,10^{-8}$

the $c_n^{(2)}$ coefficients goes rapidly to zero. As seen from Table I, the $c_n^{(2)}$ contribution to c_n can be neglected for large n. A similar situation can be observed in the case of the $\Gamma_n^{(2)}$ coefficients (see Table II).

In Table III we test the validity of the summation rules (70) for $i=1, 2$ and (71) for $j=1$, $i=1, 2$ for the ground and first excited states. The coefficients $c_n^{(i)}$ and $\Gamma_n^{(i)}$ were obtained by the method described above. The results confirm correctness of the computed coefficients. Since the coefficients $\Gamma_n^{(2)}$ go down with increasing n more rapidly than the $\Gamma_n^{(1)}$ coefficients, the accuracy of the summation rules for $i=2$ is better than that for $i=1$.

We showed in Section IIIB that if the dispersion relation (34)–(36) is correct then the series

$$-E_R^{(1)}(\kappa) = -\sum_{n=0}^{\infty} c_n^{(1)} \kappa^n, \tag{77}$$

where $c_n^{(1)}$ is given by Eq. (76), has to be the Stieltjes series. Therefore, this series must be summable by means of the Padé approximants P_N^N and P_{N+1}^N, which must form monotonically increasing lower bounds and monotonically decreasing upper

TABLE II

The $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$ Parts of the Strong Coupling Coefficients $\Gamma_n = \Gamma_n^{(1)} + \Gamma_n^{(2)}$ for the Ground and First Excited States of the Quartic Oscillator

	Ground state		First excited state	
n	$\Gamma_n^{(1)}$	$\Gamma_n^{(2)}$	$\Gamma_n^{(1)}$	$\Gamma_n^{(2)}$
0	$-0.540\,199\,498\,347\,10^{-2}$	$0.740\,616\,005\,314\,10^{0}$	$-0.852\,508\,237\,506\,10^{-1}$	$0.271\,979\,695\,780\,10^{1}$
1	$-0.239\,492\,398\,304\,10^{-2}$	$0.279\,450\,596\,862\,10^{0}$	$-0.393\,527\,585\,693\,10^{-1}$	$0.461\,511\,429\,715\,10^{0}$
2	$-0.110\,237\,579\,337\,10^{-2}$	$-0.100\,765\,214\,162\,10^{-1}$	$-0.190\,127\,417\,537\,10^{-1}$	$-0.127\,169\,704\,139\,10^{-1}$
3	$-0.528\,076\,418\,651\,10^{-3}$	$0.619\,271\,070\,696\,10^{-4}$	$-0.963\,050\,292\,490\,10^{-2}$	$-0.220\,435\,041\,057\,10^{-2}$
4	$-0.263\,549\,787\,842\,10^{-3}$	$-0.298\,944\,474\,862\,10^{-4}$	$-0.511\,252\,074\,120\,10^{-2}$	$-0.547\,685\,367\,643\,10^{-3}$
5	$-0.136\,987\,719\,905\,10^{-3}$	$-0.110\,775\,369\,023\,10^{-4}$	$-0.283\,803\,020\,978\,10^{-2}$	$-0.146\,413\,112\,022\,10^{-3}$
6	$-0.740\,361\,957\,686\,10^{-4}$	$-0.287\,924\,722\,765\,10^{-5}$	$-0.164\,133\,157\,574\,10^{-2}$	$-0.406\,288\,033\,548\,10^{-4}$
7	$-0.414\,967\,955\,613\,10^{-4}$	$-0.823\,823\,087\,432\,10^{-6}$	$-0.984\,580\,002\,966\,10^{-3}$	$-0.115\,822\,955\,812\,10^{-4}$
8	$-0.240\,440\,658\,598\,10^{-4}$	$-0.241\,728\,202\,262\,10^{-6}$	$-0.609\,813\,782\,519\,10^{-3}$	$-0.336\,971\,855\,729\,10^{-5}$
9	$-0.143\,534\,032\,367\,10^{-4}$	$-0.716\,547\,139\,744\,10^{-7}$	$-0.388\,303\,025\,097\,10^{-3}$	$-0.995\,985\,741\,434\,10^{-6}$
10	$-0.879\,886\,184\,802\,10^{-5}$	$-0.214\,961\,124\,549\,10^{-7}$	$-0.253\,233\,814\,706\,10^{-3}$	$-0.298\,110\,391\,784\,10^{-6}$
11	$-0.552\,216\,586\,747\,10^{-5}$	$-0.651\,262\,905\,675\,10^{-8}$	$-0.168\,594\,304\,846\,10^{-3}$	$-0.901\,466\,298\,289\,10^{-7}$
12	$-0.353\,866\,997\,409\,10^{-5}$	$-0.198\,905\,498\,971\,10^{-8}$	$-0.114\,277\,089\,713\,10^{-3}$	$-0.274\,922\,058\,436\,10^{-7}$
13	$-0.230\,998\,892\,364\,10^{-5}$	$-0.611\,659\,701\,702\,10^{-9}$	$-0.786\,861\,325\,068\,10^{-4}$	$-0.844\,450\,419\,054\,10^{-8}$
14	$-0.153\,304\,660\,923\,10^{-5}$	$-0.189\,201\,959\,884\,10^{-9}$	$-0.549\,359\,742\,123\,10^{-4}$	$-0.260\,967\,700\,929\,10^{-8}$
15	$-0.103\,261\,941\,769\,10^{-5}$	$-0.588\,237\,232\,116\,10^{-10}$	$-0.388\,301\,748\,965\,10^{-4}$	$-0.810\,744\,122\,325\,10^{-9}$
16	$-0.704\,915\,346\,102\,10^{-6}$	$-0.183\,701\,631\,008\,10^{-10}$	$-0.277\,511\,116\,507\,10^{-4}$	$-0.253\,029\,511\,575\,10^{-9}$
17	$-0.487\,088\,983\,367\,10^{-6}$	$-0.575\,941\,322\,140\,10^{-11}$	$-0.200\,319\,363\,661\,10^{-4}$	$-0.792\,880\,415\,186\,10^{-10}$
18	$-0.340\,323\,836\,657\,10^{-6}$	$-0.181\,199\,564\,332\,10^{-11}$	$-0.145\,915\,322\,929\,10^{-4}$	$-0.249\,340\,723\,931\,10^{-10}$
19	$-0.240\,209\,069\,512\,10^{-6}$	$-0.571\,858\,764\,418\,10^{-12}$	$-0.107\,170\,555\,243\,10^{-4}$	$-0.786\,610\,796\,391\,10^{-11}$
20	$-0.171\,139\,892\,888\,10^{-6}$	$-0.180\,982\,386\,906\,10^{-12}$	$-0.793\,148\,447\,724\,10^{-5}$	$-0.248\,866\,035\,673\,10^{-11}$

bounds to the value of $E_R^{(1)}(\kappa)$ (see, for example, [3, 17, 18]). The sum $\sum_{n=0}^{\infty} c_n^{(2)}\kappa^n$ is a rapidly convergent series (see Table I) with the radius of convergence $R_2 = 2$ (see the estimate (52)). Thus, the quantities $L_N = P_N^N + \sum_{n=0}^{\infty} c_n^{(2)}\kappa^n$ and $U_N = P_{N+1}^N + \sum_{n=0}^{\infty} c_n^{(2)}\kappa^n$ have to be monotonically increasing lower bounds and monotonically decreasing upper bounds to the exact value of the energy $E_R(\kappa)$. In Tables IV and V, these lower and upper bounds are compared with the exact values

TABLE III

Test of Validity of the Summation Rules for the Ground and First Excited States of the Quartic Oscillator

	Ground state		First excited state	
i	$\sum_{n=0}^{50} \Gamma_n^{(i)} - c_0^{(i)}$	$\sum_{n=1}^{50} n\Gamma_n^{(i)} - c_1^{(i)}$	$\sum_{n=0}^{50} \Gamma_n^{(i)} - c_0^{(i)}$	$\sum_{n=1}^{50} n\Gamma_n^{(i)} - c_1^{(i)}$
1	$-0.347\,10^{-9}$	$0.194\,10^{-7}$	$-0.343\,10^{-7}$	$0.192\,10^{-5}$
2	$-0.169\,10^{-27}$	$0.870\,10^{-26}$	$0.103\,10^{-24}$	$0.722\,10^{-25}$

Note. The first and third columns correspond to Eq. (70) and the second and fourth columns correspond to Eq. (71) for $j = 1$. Here, the contributions from the axis ($i = 1$) and the circle ($i = 2$) to the Γ_n coefficients are considered separately.

TABLE IV

The Increasing Lower Bounds and Decreasing Upper Bounds $L_N = P_N^N + \sum_{n=0}^{100} c_n^{(2)}$ and $U_N = P_{N+1}^N + \sum_{n=0}^{100} c_n^{(2)}$ to the Exact Value of the Renormalized Ground State Energy $E_R(\kappa = 1) = \Gamma_0 = 0.735\,214\,010\,331\,216\ldots$ of the Quartic Oscillator

N	L_N	U_N
1	0.734 946 792 144	0.735 233 563 644
2	0.735 184 272 374	0.735 216 752 944
3	0.735 209 350 358	0.735 214 505 988
4	0.735 213 100 750	0.735 214 117 117
5	0.735 213 802 749	0.735 214 036 556
6	0.735 213 956 993	0.735 214 017 466
7	0.735 213 995 276	0.735 214 012 440
8	0.735 214 005 742	0.735 214 010 999
9	0.735 214 008 839	0.735 214 010 555
10	0.735 214 009 819	0.735 214 010 410
11	0.735 214 010 146	0.735 214 010 360
12	0.735 214 010 261	0.735 214 010 342
13	0.735 214 010 304	0.735 214 010 335
14	0.735 214 010 320	0.735 214 010 333
15	0.735 214 010 326	0.735 214 010 331
16	0.735 214 010 329	0.735 214 010 331
17	0.735 214 010 330	0.735 214 010 331
18	0.735 214 010 330	0.735 214 010 331
19	0.735 214 010 331	0.735 214 010 331
20	0.735 214 010 331	0.735 214 010 331

Note. P_N^N and P_{N+1}^N are the Padé approximants to the series $\sum_n c_n^{(1)}$.

of $E_R(\kappa)$ for $\kappa = 1$. Since the lower and upper bounds L_N and U_N converge rapidly to the exact value of the energy $E_R(\kappa = 1)$, it is obvious that even better convergence of L_N and U_N to the exact value of the energy $E_R(\kappa)$ is obtained for $0 < \kappa < 1$. These results confirm our conclusion made in Section IIIB that the energy $-E_R^{(1)}(\kappa)$ is the Stieltjes function. This, together with the method of computing the $c_n^{(1)}$ coefficients from Eq. (76), confirms the analyticity of $E_R(\kappa)$ in the domain $0 < |\kappa| < R_2$, $\arg \kappa \in (-\pi, \pi)$.

VIII. CONCLUSIONS

In this paper, a detailed investigation of the weak coupling and strong coupling perturbative expansions of the renormalized energy of the quartic oscillator was performed and the most important questions of this perturbation theory were clarified.

First, known results for the ordinary energy $E(\beta)$ and the weak coupling expansion coefficients b_n were briefly summarized.

TABLE V

The Increasing Lower Bounds and Decreasing Upper Bounds $L_N = P_N^N + \sum_{n=0}^{100} c_n^{(2)}$ and $U_N = P_{N+1}^N + \sum_{n=0}^{100} c_n^{(2)}$ to the Exact Value of the Renormalized First Excited State Energy $E_R(\kappa = 1) = \Gamma_0 = 2.634\,546\,134\,058\,831\ldots$ of the Quartic Oscillator

N	L_N	U_N
1	2.267 724 053 339	2.635 115 596 244
2	2.633 639 111 165	2.634 636 440 943
3	2.634 387 618 437	2.634 563 889 426
4	2.634 512 742 628	2.634 550 205 814
5	2.634 538 064 187	2.634 547 183 933
6	2.634 543 964 726	2.634 546 431 169
7	2.634 545 498 786	2.634 546 224 799
8	2.634 545 934 382	2.634 546 163 601
9	2.634 546 067 430	2.634 546 144 216
10	2.634 546 110 654	2.634 546 137 720
11	2.634 546 125 461	2.634 546 135 434
12	2.634 546 130 773	2.634 546 134 595
13	2.634 546 132 758	2.634 546 134 275
14	2.634 546 133 527	2.634 546 134 148
15	2.634 546 133 835	2.634 546 134 097
16	2.634 546 133 962	2.634 546 134 075
17	2.634 546 134 016	2.634 546 134 066
18	2.634 546 134 039	2.634 546 134 062
19	2.634 546 134 049	2.634 546 134 060
20	2.634 546 134 054	2.634 546 134 059

Note. P_N^N and P_{N+1}^N are the Padé approximants to the series $\sum_n c_n^{(1)}$.

Further, the analytic structure of the renormalized energy $E_R(\kappa)$ was investigated. In contrast to $E(\beta)$, the renormalized energy $E_R(\kappa)$ is not analytic in the whole sheet $\arg \kappa \in (-\pi, \pi)$. However, we showed that the renormalized energy $E_R(\kappa)$ is analytic in the sheet $\arg \kappa \in (-\pi, \pi)$ at least in the region $0 < |\kappa| < \kappa_{\max}$, where $\kappa_{\max} \geqslant 8.321$ for all the states of the quartic oscillator. Outside this region non-analytic behavior of $E_R(\kappa)$ related to the branch points of $E(\beta)$ can be expected. From this analytic structure, we showed that the strong coupling expansion (10) converges for all $\kappa \in (0, 1]$ corresponding to the anharmonic oscillator with $\beta \in (0, \infty)$ and for all $\kappa \in [1, 2)$ corresponding to the double-well problem. Further, we found the exact dispersion relation for the energy $E_R(\kappa) = E_R^{(1)}(\kappa) + E_R^{(2)}(\kappa)$ consisting of two parts. The first part, taken with the negative sign, is the Stieltjes function. Dispersion relations with a similar structure were found also for the weak and strong coupling expansion coefficients c_n and Γ_n. These exact equations for the c_n and Γ_n coefficients lead to the analytic large-order formulas known from previous papers [2, 5, 7, 8, 11]. In contrast to these approaches, our derivation of the large-order formulas is a new, more rigorous approach following from the analytic structure of the renormalized energy $E_R(\kappa)$. We derived also the summation rules giving the mutual relation of the Stieltjes and non-Stieltjes parts of the c_n and Γ_n

coefficients. Detailed numerical tests performed for the ground and first excited states confirm the correctness of our conclusions.

Now we compare the weak and strong coupling expansions of $E(\beta)$ and $E_R(\kappa)=(1-\kappa)^{1/2}E(\beta)$. The weak coupling expansions $E(\beta)=\sum_n b_n\beta^n$ and $E_R(\kappa)=\sum_n c_n\kappa^n$ are expanded at the singular points $\beta=0$ and $\kappa=0$, respectively. From the physical point of view, these points are singular since the Hamiltonians $H=p^2+x^2+\beta x^4$ and $H_R=p^2+x^2+\kappa(x^4/3-x^2)$ do not have bound states for $\beta<0$ and $\kappa<0$ and the energies $E(\beta)$ and $E_R(\kappa)$ are not analytic at the points $\beta=0$ and $\kappa=0$. Therefore, these expansions are not convergent Taylor series and diverge for arbitrary $\beta\in(0,\infty)$ and corresponding $\kappa\in(0,1)$ (compare with the well-known Dyson argument in quantum electrodynamics [22]). In contrast to these divergent expansions, the renormalized strong coupling expansion $E_R(\kappa)=\sum_n \Gamma_n(1-\kappa)^n$ has different properties.

It follows from the analytic structure of $E_R(\kappa)$ that $E_R(\kappa)$ is analytic in the circle $|1-\kappa|<1$ in the complex κ-plane. Therefore, the series $E_R(\kappa)=\sum_n \Gamma_n(1-\kappa)^n$ is the convergent Taylor series for all $|1-\kappa|<1$. From a physical point of view, the Hamiltonian $H_R=p^2+x^4/3+(1-\kappa)(x^2-x^4/3)$ becomes for $1-\kappa<0$ the Hamiltonian of the double-well problem which has bound states and the energy $E_R(\kappa)$ can be analytic at the point $\kappa=1$.

The strong coupling expansion of the ordinary energy $E(\beta)=\beta^{1/3}\sum_n K_n\beta^{-2n/3}$ is the expansion at the point $\beta=\infty$ corresponding to $\kappa=1$. Therefore, it is analogous to the strong coupling expansion of the renormalized energy $E_R(\kappa)=\sum_n \Gamma_n(1-\kappa)^n$. However, because of the prefactor $\beta^{1/3}$, which goes to zero for $\beta\to 0$, the expansion $\sum_n K_n\beta^{-2n/3}$ must diverge for $\beta\to 0$ when $E(0)=2K+1$. From the point of view of the analytic structure of the energy $E(\beta)$, this expansion diverges because of the existence of the square-root branch points on the three-sheeted Riemann surface near the origin $\beta=0$. Therefore, this expansion has properties less favorable than those of the expansion $E_R(\kappa)=\sum_n \Gamma_n(1-\kappa)^n$, which converges for all $\kappa\in(0,2)$.

We can conclude that the renormalized strong coupling expansion $E_R(\kappa)=\sum_n \Gamma_n(1-\kappa)^n$ is the most advantageous perturbative approach to the anharmonic oscillators. We believe that the results of this paper contribute not only to deeper understanding of the perturbation theories for the anharmonic oscillators and double-well problems but to better understanding of behavior of the perturbation theories at large orders in general.

APPENDIX A

Equations (48) and (29) lead to

$$c_n^{(2)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\left(\frac{R_2e^{i\varphi}}{3}\right)^{1/3}\sum_{j=0}^{\infty}K_j\left(\frac{3^{2/3}(1-R_2e^{i\varphi})}{(R_2e^{i\varphi})^{2/3}}\right)^j(R_2e^{i\varphi})^{-n}$$
$$=\frac{R_2^{1/3-n}}{2\pi 3^{1/3}}\sum_{j=0}^{\infty}K_j\left(\frac{3}{R_2}\right)^{2j/3}I(n,j), \tag{78}$$

where

$$I(n, j) = \int_{-\pi}^{\pi} d\varphi\, e^{i\varphi((1-2j)/3-n)}(1 - R_2 e^{i\varphi})^j. \tag{79}$$

Using the binomial formula we get

$$I(n, j) = \sum_{l=0}^{j} \frac{j!}{l!(j-l)!} (-R_2)^l \int_{-\pi}^{\pi} d\varphi\, e^{i\varphi((1-2j)/3-n+l)}. \tag{80}$$

For $j \neq 3q+2$, integration of this equation yields

$$I(n, j) = (-1)^n \frac{2 \sin\left(\frac{1-2j}{3}\pi\right)}{\frac{1-2j}{3} - n + l} \sum_{l=0}^{j} \frac{j!}{l!(j-l)!} R_2^l. \tag{81}$$

For $j = 3q+2$ the integral in Eq. (81) equals zero except for the case $l = n+2q+1$ and $q \geqslant n-1$ when

$$I(n, 3q+2) = \frac{2\pi(3q+2)!\,(-R_2)^{n+2q+1}}{(n+2q+1)!\,(q-n+1)!}. \tag{82}$$

APPENDIX B

Equations (59) and (29) lead to

$$\Gamma_n^{(2)} = \frac{(-1)^{n+j}}{2\pi} \sum_{j=0}^{\infty} 3^{(2j-1)/3} K_j I(n, j), \tag{83}$$

where

$$I(n, j) = \int_{-\pi}^{\pi} d\varphi\, (R_2 e^{i\varphi} + 1)^{(1-2j)/3} (R_2 e^{i\varphi})^{j-n}. \tag{84}$$

To calculate the last integral we use the Gauss theorem

$$\oint_C dz\, f(z) = 0 \tag{85}$$

for the function

$$f(z) = (z+1)^{(1-2j)/3}\, z^{j-n-1}. \tag{86}$$

We consider two cases:

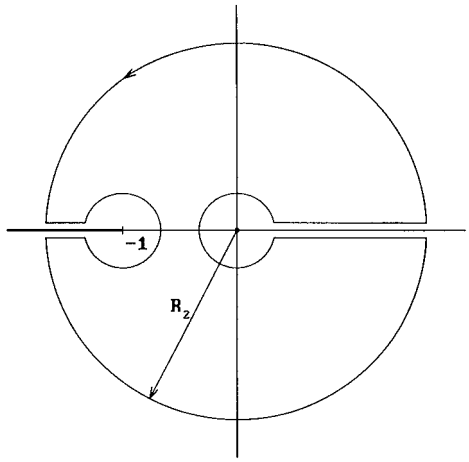

FIG. 3. Integration path in the complex z plane. The radii of small circles equal ε.

1. $j \geqslant 3n-1$. Using the integration path C shown in Fig. 3, Eq. (85) can be written in the form

$$I(n, j) = (-1)^{j-n}\, 2 \sin\left(\frac{1-2j}{3}\,\pi\right) \int_{1+\varepsilon}^{R_2} dz (z-1)^{(1-2j)/3}\, z^{j-n-1} + \int_{-\pi}^{\pi} d\varphi (\varepsilon e^{i\varphi})^{(1-2j)/3+1} (-1+\varepsilon e^{i\varphi})^{j-n-1} + \int_{0}^{2\pi} d\varphi (\varepsilon e^{i\varphi})^{j-n}, \quad (87)$$

where $\varepsilon \to 0+$. Now we consider three cases:

(a) $j > n$ and $j \neq 3q+2$. In this case, the last integral in Eq. (87) equals zero. The substitution

$$u = (z-1)^{1/3} \quad (88)$$

and the use of the binomial formula in the first integral in Eq. (87) yields

$$I(n, j) = (-1)^{j-n}\, 2 \sin\left(\frac{1-2j}{3}\,\pi\right) \sum_{l=0}^{j-n-1} \frac{(j-n-1)!}{l!(j-n-1-l)!} \frac{(R_2-1)^{(1-2j)/3+l+1}}{(1-2j)/3+l+1}. \quad (89)$$

Here, the contribution of the lower bound of the first integral in Eq. (87) cancels with the second integral in Eq. (87).

(b) $j = 3q+2$. In this case, the first and third integrals in Eq. (87) equal zero. Using the binomial formula in the remaining integral we get

$$I(n, j) = \sum_{l=0}^{3q+1-n} (-1)^{3q+1-n+l}\, \varepsilon^{-2q+l} \frac{(j-n-1)!}{l!(j-n-1-l)!} \int_{-\pi}^{\pi} d\varphi\, e^{i\varphi(l-2q)}. \quad (90)$$

The last integral equals zero except for the case $l=2q$ and $q \geqslant n-1$ when

$$I(n, 3q+2) = \frac{2\pi(-1)^{5q+1-n}(j-n-1)!}{2q!(j-n-1-2q)!}. \tag{91}$$

(c) $j < n+1$. In this case we get $j=0$ and $n=0$. Then, the second integral in Eq. (87) equals zero. Substituting Eq. (88) into the first integral in Eq. (87) we get

$$I(0, 0) = 2\pi + 3^{3/2}(R_2-1)^{1/3} - 3^{3/2}\int_0^{(R_2-1)^{1/3}} \frac{du}{u^3+1}. \tag{92}$$

The last integral can be calculated easily.

2. $j < 3n-1$. Using the Gauss theorem (85) for the function (86) with the integration path C shown in Fig. 4 we get

$$I(n, j) = (-1)^{j-n-1}\, 2\sin\left(\frac{1-2j}{3}\pi\right)\int_{R_2}^{\infty} dz(z-1)^{(1-2j)/3}\, z^{j-n-1}$$

$$+ \lim_{R_3 \to \infty} \int_{-\pi}^{\pi} d\varphi(R_3 e^{i\varphi}+1)^{(1-2j)/3}\,(R_3 e^{i\varphi})^{j-n}.$$

The last integral equals zero. The substitution (88) yields

$$I(n, j) = (-1)^{j-n-1}\, 6\sin\left(\frac{1-2j}{3}\pi\right) + \int_{(R_2-1)^{1/3}}^{\infty} du\, u^{3-2j}(u^3+1)^{j-n-1}. \tag{94}$$

Again, the last integral can be easily calculated.

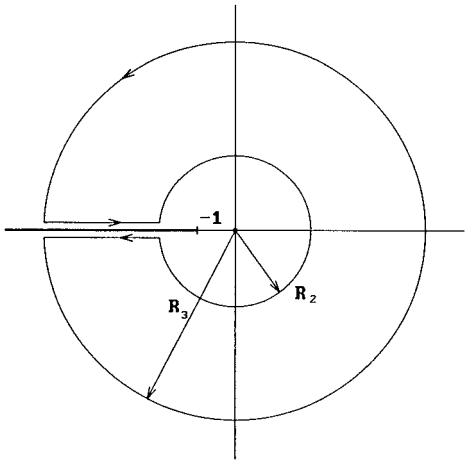

FIG. 4. Integration path in the complex z-plane.

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