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Algebraic approach to non-separable two-dimensional Schrödinger equation: Ground states for polynomial and Morse-like potentials

Research Article

Vladimír Tichý[1]*, Lubomír Skála[2], René Hudec[1]

1 *Czech Technical University in Prague, Faculty of Electrical Engineering, Technická 2, 16627 Prague 6, Czech Republic*

2 *Charles University, Faculty of Mathematics and Physics, Ke Karlovu 3, 12116 Prague 2, Czech Republic*

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Abstract: This paper presents a direct algebraic method of searching for analytic solutions of the two-dimensional time-independent Schrödinger equation that is impossible to separate into two one-dimensional ones. As examples, two-dimensional polynomial and Morse-like potentials are discussed. Analytic formulas for the ground state wave functions and the corresponding energies are presented. These results cannot be obtained by another known method.

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1. Introduction

There are many types of potentials for which the one-dimensional Schrödinger equation is analytically solvable. Usually, the two-dimensional Schrödinger equation is solved by some type of separation of variables (see e.g. [1–5]). If this method fails, no general approach to solving the Schrödinger equation is known.

In this paper, a method of searching for the solutions of the two-dimensional Schrödinger equation if separation of variables is not applicable is presented. The presented method is based on the algebraic approach to solving one-dimensional problems presented in [6–8]. The method was generalized to two-dimensions in papers [9–11], where the potential was assumed in the form of a polynomial in variables x, y. In this paper, we present a generalized approach to searching for analytic solutions of the Schrödinger equation. The method is applied to the two-dimensional polynomial and Morse potentials.

Polynomial and Morse-like potentials are very useful in various applications in physics and chemistry. Exact analytic solutions for at least some states of the two-dimensional Schrödinger equation would be very useful not only for testing approximate methods of solutions but also for extending our knowledge of analytic one-dimensional solutions to more dimensional ones.

*E-mail: vladimir-tichy@email.cz

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First, we summarize our method for the one-dimensional problems [6–8]. In these papers, it is assumed that solutions $\psi(x)$ of the one-dimensional Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \tag{1}$$

are linear combinations of the functions $\psi_m(x)$ in the form

$$\psi(x) = \sum_m d_m \psi_m(x), \tag{2}$$

where

$$\psi_m(x) = f^m(x)h(x). \tag{3}$$

$f^m(x)$ denotes m-th power of $f(x)$. The potential V is assumed in the form

$$V(x) = \sum_m V_m f^m(x). \tag{4}$$

Function $h(x)$ is equal to the ground state wave function and is searched in the form

$$h(x) = \exp\left(-\int \sum_m h_m f^m(x) dx\right). \tag{5}$$

Using this approach, it is possible to take different forms of the function $f(x)$ and to test if the Schrödinger equation with the potential V of the form (4) has analytic solutions obeying the corresponding boundary conditions. In Eq. (4), limits for index m are given by the potential and if necessary, negative values of m can also be included. After substituting Eqs. (2-5) into the Eq. (1), a system of algebraic equations for unknowns d_m and h_m is obtained. Possible values of indices m in Eqs. (2) and (5) follow from the condition that the number of the resulting equations has to be equal to the number of unknowns.

2. Generalization to two-dimensions

In [9], first attempt to generalize this method to two dimensions was presented. In this paper, the Schrödinger equation is assumed in the form

$$-\Delta\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y), \tag{6}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Its solutions $\psi(x,y)$ are assumed in the form

$$\psi(x,y) = \sum_{m,n} a_{mn}\psi_{mn}, \tag{7}$$

where

$$\psi_{mn}(x,y) = f^m(x)g^n(y)h(x,y). \tag{8}$$

The potential $V(x,y)$ can be assumed in a general form

$$V(x,y) = \sum_{m,n} V_{mn} f^m(x)g^n(y). \tag{9}$$

In this paper, we restrict ourselves to the case

$$V(x,y) = \sum_{m\geq 0\ n\geq 0}^{m+n\leq 2M} V_{mn} f^m(x)g^n(y). \tag{10}$$

In our previous papers [9–11], a general formula for the two-dimensional function h was missing. Only the simplest cases with $f(x) = x$, $f(y) = y$ were discussed and the assumption

$$h(x,y) = \exp\left(\sum d_{ij}x^i y^j\right) \tag{11}$$

for the function h was used. In [10], also the $\mathcal{PT}$-symmetric solution was discussed, however, the method was restricted to the polynomial potentials, too.

3. Ground state wave function

In this paper, the function $h(x,y)$ is assumed in the form

$$h(x,y) = \exp\left[-J(x,y)\right], \tag{12}$$

where the function $J(x,y)$ is to be found.
The function $h(x,y)$ is supposed to be equal to the ground state wave function of the Schrödinger equation (6), as in all previously presented one-dimensional as well as two-dimensional cases. Therefore, it should fulfil the Schrödinger equation

$$-\Delta h(x,y) + V(x,y)h(x,y) = E_0 h(x,y). \tag{13}$$

Substituting Eq. (12) into Eq. (13), we obtain a partial differential equation (PDE) for the function $J(x,y)$

$$\left[\frac{\partial J(x,y)}{\partial x}\right]^2 + \left[\frac{\partial J(x,y)}{\partial y}\right]^2 - \frac{\partial^2 J(x,y)}{\partial x^2} - \frac{\partial^2 J(x,y)}{\partial y^2} = V(x,y) - E_0. \tag{14}$$

The potential V on the right-hand side of Eq. (14) is supposed in the form of polynomial in the functions f, g in

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Eq. (10). Therefore, the partial derivatives of $J(x,y)$ are searched in a similar form

$$\frac{\partial J(x,y)}{\partial x} = \sum_{m\geq 0\ n\geq 0}^{m+n\leq M} c_{mn} f^m(x) g^n(y), \quad (15)$$

$$\frac{\partial J(x,y)}{\partial y} = \sum_{m\geq 0\ n\geq 0}^{m+n\leq M} d_{mn} f^m(x) g^n(y). \quad (16)$$

The limits of sums are given by the condition that it is necessary to obtain the same order of the polynomial at both sides of Eq. (13). A general discussion is difficult, so the following calculations are made for specific values of M. The solutions are searched for in the following steps:

- The value of M and suitable functions f, g are taken. In this paper, functions $f(x) = x$, $g(y) = y$ or $f(x) = e^{-x}$, $g(y) = e^{-y}$ are used.
- For given functions f, g, conditions for the coefficients c_{ij}, d_{ij} are found in such a way that the system of PDE (15)-(16) is solvable.
- The function J solving the system of Eqs. (15)-(16) is found. Function J is substituted into Eq. (12).
- Equation (12) is substituted into the Schrödinger Equation (13). Comparing the terms of the same order, a system of algebraic equations for unknown coefficients c_{ij}, d_{ij} is obtained.
- Finally, formulas for the coefficients c_{ij}, d_{ij} and constrains for the coefficients V_{ij} are found.

3.1. Case M = 1

For $M = 1$, Eq. (10) equals

$$V(x,y) = V_{20} f^2(x) + V_{02} g^2(y) + V_{11} f(x) g(y) + V_{10} f(x) + V_{01} g(x) \quad (17)$$

and Eqs. (15)-(16) have the form

$$\frac{\partial J(x,y)}{\partial x} = c_{00} + c_{10} f(x) + c_{01} g(y), \quad (18)$$

$$\frac{\partial J(x,y)}{\partial y} = d_{00} + d_{10} f(x) + d_{01} g(y). \quad (19)$$

Calculating the derivative of Eq. (18) with respect to y and Eq. (19) with respect to x and comparing the right-hand sides of the resulting equations, the condition of solvability of the system of PDE (18)-(19) equals

$$d_{10} \frac{\mathrm{d}f(x)}{\mathrm{d}x} = c_{01} \frac{\mathrm{d}g(y)}{\mathrm{d}y}. \quad (20)$$

Equation (20) could be fulfilled taking $c_{01} = d_{10} = 0$. However, in this case all resulting solutions can be obtained also by separation of variables, so they are not interesting. Another option is to put $f(x) = x$, $g(y) = y$ and $c_{01} = d_{10}$. However, it leads to the well-known problem of the two-dimensional harmonic oscillator. The same result is obtained for general linear functions $f(x) = f_1 x + f_2$ and $g(y) = g_1 y + g_2$.

3.2. Case M = 2

For $M = 2$, the situation is completely different. In this case, Eqs. (15)-(16) have the form

$$\frac{\partial J(x,y)}{\partial x} = c_{00} + c_{10} f + c_{01} g + c_{20} f^2 + c_{02} g^2 + c_{11} f g, \quad (21)$$

$$\frac{\partial J(x,y)}{\partial y} = d_{00} + d_{10} f + d_{01} g + d_{20} f^2 + d_{02} g^2 + d_{11} f g. \quad (22)$$

Condition of solvability of PDE (21)-(22) is now

$$(d_{10} + 2\, d_{20} f + d_{11} g) \frac{\mathrm{d}f}{\mathrm{d}x} = (c_{01} + 2\, c_{02} g + c_{11} f) \frac{\mathrm{d}g}{\mathrm{d}y}. \quad (23)$$

For given functions f, g, Eq. (23) represents constraints for coefficients c_{ij} and d_{ij}.

In this paper, we aim to simple cases when Eq. (23) is easy to fulfil. First case is $\frac{\mathrm{d}f}{\mathrm{d}x} = \mathrm{const}$, $\frac{\mathrm{d}g}{\mathrm{d}y} = \mathrm{const}$, i.e. $f(x)$ and $g(y)$ are linear functions. Without loss of generality, functions $f(x) = x$, $g(y) = y$ were chosen. Second presented case is $\frac{\mathrm{d}f}{\mathrm{d}x} = f$, $\frac{\mathrm{d}g}{\mathrm{d}y} = g$, i.e. $f = e^x$, $g = e^y$. There could be some constants included but without loss of generality, they were omitted.

3.2.1. Two-dimensional quartic polynomial potential

If $f(x) = x$, $g(y) = y$ is chosen, then Eq. (23) equals

$$d_{10} + 2\, d_{20} x + d_{11} y = c_{01} + 2\, c_{02} y + c_{11} x. \quad (24)$$

Consequently, three constrains are gotten: $c_{01} = d_{10}$, $c_{11} = 2 d_{20}$ and $2 c_{02} = d_{11}$. The system of PDE (21)-(22) has the solution

$$J = \frac{c_{20}}{3} x^3 + \frac{d_{02}}{3} y^3 + d_{20} x^2 y + c_{02} x y^2 + \frac{c_{10}}{2} x^2 + \frac{d_{01}}{2} y^2 + c_{01} x y + c_{00} x + d_{00} y. \quad (25)$$

Here, an irrelevant integration constant is omitted. Substituting Eq. (25) into (12), an equation which is equivalent to Eq. (11) is obtained. This result justifies Eq. (11) used in the previous papers [9–11] as the assumption. The complete discussion and results for the quartic polynomial potentials are given in these papers.

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3.2.2. Two-dimensional quartic Morse potential

If $f(x) = e^x$, $g(y) = e^y$ is chosen, then Eq. (23) equals

$$d_{10} f + 2\, d_{20} f^2 + d_{11} f g = c_{01} g + 2\, c_{02} g^2 + c_{11} f g. \quad (26)$$

Now the constrain $d_{11} = c_{11}$ is seen and also it is seen that coefficients c_{02}, c_{01}, d_{20}, d_{10} have to equal to zero. Eqs. (21)-(22) contain totally 12 parameters c_{ij}, d_{ij}. Four of them have to equal to zero and two of them are related by the constrain. Therefore, seven of the coefficients c_{ij}, d_{ij} remain independent.
Substituting these conditions into Eqs. (21)-(22), system of PDE for function J is obtained

$$\begin{aligned} \frac{\partial J(x,y)}{\partial x} &= c_{00} + c_{10} e^{-x} + c_{20} e^{-2x} + c_{11} e^{-x-y}, \quad (27) \\ \frac{\partial J(x,y)}{\partial y} &= d_{00} + d_{01} e^{-y} + d_{02} e^{-2y} + c_{11} e^{-x-y}. \quad (28) \end{aligned}$$

System of PDE (27)-(28) has the solution

$$\begin{aligned} J(x,y) &= d_{00} y - d_{01} e^{-y} - \frac{d_{02}}{2} e^{-2y} - c_{11} e^{-x-y} \\ &- \frac{c_{20}}{2} e^{-2x} - c_{10} e^{-x} + c_{00} x. \end{aligned} \quad (29)$$

Here, an irrelevant integration constant is omitted. Substituting Eq. (29) into Eq. (12), the formula for the function h is obtained as

$$h(x,y) = \exp\left(\frac{c_{20}}{2} e^{-2x} + \frac{d_{02}}{2} e^{-2y} + c_{11} e^{-x-y} + c_{10} e^{-x} + d_{01} e^{-y} - c_{00} x - d_{00} y \right). \quad (30)$$

Substituting $M = 2$, $f(x) = e^{-x}$ and $f(y) = e^{-y}$ into Eq. (10), the formula for the potential is

$$V(x,y) = \sum_{m\geq 0\ n\geq 0}^{m+n\leq 4} V_{mn} e^{-mx-ny}. \quad (31)$$

Without loss of generality, it is assumed that $V_{00} = 0$. Substituting Eqs. (30)-(31) into Eq. (13), the following equation is obtained

$$\begin{aligned} & {c_{20}}^2 e^{-4x} + 2\, c_{20} c_{10} e^{-3x} + ({c_{10}}^2 + 2\, c_{20} c_{00} + 2\, c_{20}) e^{-2x} + (2\, c_{10} c_{00} + c_{10}) e^{-x} + {d_{02}}^2 e^{-4y} + 2\, d_{02} d_{01} e^{-3y} \\ & + ({d_{01}}^2 + 2\, d_{02} d_{00} + 2\, d_{02}) e^{-2y} + (2\, d_{01} d_{00} + d_{10}) e^{-y} + 2\, {c_{11}}^2 e^{-2(x+y)} + 2 c_{11} (c_{00} + d_{00} + 1) e^{-x-y} \\ & + 2\, c_{20} c_{11} e^{-3x-y} + 2\, c_{11} c_{10} e^{-2x-y} + 2\, d_{02} c_{11} e^{-x-3y} + 2\, c_{11} d_{01} e^{-x-2y} + {c_{00}}^2 + {d_{00}}^2 \quad = \sum_{m\geq 0\ n\geq 0}^{m+n\leq 4} V_{mn} e^{-mx-ny} - E_0. \end{aligned} \quad (32)$$

The consequent calculations are made in a similar manner as it has been done in papers [9–11] and they are not presented in detail here. Comparing the terms of the same order, a system of 15 algebraic equations is obtained. One of them represents the formula for the ground state energy. Ideally, the remaining 14 equations would be transformed to 7 equations for 7 coefficients c_{ij}, d_{ij} in dependence on 14 parameters V_{ij} of the potential. Parameters V_{ij} are not independent but they have to fulfil seven constrains.
As a simpler way to express the results, parametrization by 7 independent real parameters $W_{40} = \pm\sqrt{V_{40}}$, $W_{04} = \pm\sqrt{V_{40}}$, $W_{22} = \pm\sqrt{V_{22}}$, V_{30}, V_{03}, $\alpha = c_{00}$, $\beta = d_{00}$ was found. Used parametrization ensures that all constrains ale fulfilled.
As a result, the analytic solution of the Schrödinger equation (6) for the potential is obtained as

$$\begin{aligned} V(x,y) &= {W_{40}}^2 e^{-4x} + {W_{04}}^2 e^{-4y} + \sqrt{2} W_{40} W_{22} e^{-3x-y} + \sqrt{2} W_{04} W_{22} e^{-x-3y} + {W_{22}}^2 e^{-2x-2y} + V_{30} e^{-3x} + V_{03} e^{-3y} \\ & + \frac{W_{22} V_{30}}{\sqrt{2} W_{40}} e^{-2x-y} + \frac{W_{22} V_{03}}{\sqrt{2} W_{04}} e^{-x-2y} + \frac{8 {W_{40}}^3 (\alpha+1) + {V_{30}}^2}{4 {W_{40}}^2} e^{-2x} + \frac{8 {W_{04}}^3 (\beta+1) + {V_{03}}^2}{4 {W_{04}}^2} e^{-2y} \\ & + \sqrt{2} W_{22} (\alpha + \beta + 1) e^{-x-y} + \frac{V_{30}(2\alpha+1)}{2 W_{40}} e^{-x} + \frac{V_{03}(2\beta+1)}{2 W_{04}} e^{-y}. \end{aligned} \quad (33)$$

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The corresponding ground state wave function has the form

$$\psi_0(x,y) \;=\; \exp\left(\frac{W_{40}}{2}e^{-2x}+\frac{W_{04}}{2}e^{-2y}+\frac{W_{22}}{\sqrt{2}}e^{-x-y}+\frac{V_{30}}{2W_{40}}e^{-x}+\frac{V_{03}}{2W_{04}}e^{-y}-\alpha x-\beta y\right) \tag{34}$$

and the ground state energy equals

$$E_0=-\alpha^2-\beta^2. \tag{35}$$

The potential (33) represents the two-dimensional generalisation of the quartic Morse potential

$$V(x)=\sum_{n=1}^{4}V_n\left[1-\exp(-x)\right]^n \tag{36}$$

presented in [6, 7]. The ground state wave function (34) has continuous derivatives in the entire plane (x,y). If $\alpha>0$, $\beta>0$, $W_{40}<0$, $W_{04}<0$ and $W_{22}\le 0$, the wave function (34) is also quadratically integrable in the entire plane (x,y).

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3.3. Case M = 3

For $M=3$, Eqs. (15)-(16) have the form

$$\frac{\partial J}{\partial x}=c_{00}+c_{10}f+c_{01}g+c_{20}f^2+c_{02}g^2+c_{11}fg+c_{30}f^3+c_{21}f^2g+c_{12}fg^2+c_{03}g^3, \tag{37}$$

$$\frac{\partial J}{\partial y}=d_{00}+d_{10}f+d_{01}g+d_{20}f^2+d_{02}g^2+d_{11}fg+d_{30}f^3+d_{21}f^2g+d_{12}fg^2+d_{03}g^3. \tag{38}$$

Condition of solvability of the system of PDE (37)-(38) representing constrains for the coefficients c_{ij}, d_{ij} is

$$\left(c_{01}+2c_{02}g+c_{11}f+c_{21}f^2+2c_{12}fg+3c_{03}g^2\right)\frac{\mathrm{d}g}{\mathrm{d}y} \;=\; \left(d_{10}+2d_{20}f+d_{11}g+3d_{30}f^2+2d_{21}fg+d_{12}g^2\right)\frac{\mathrm{d}f}{\mathrm{d}x}. \tag{39}$$

The following calculations can be made in a similar manner as in the case of the quartic Morse potential. For this reason, detailed calculations are not presented.

3.3.1. Two-dimensional sextic polynomial potential

In this case, it is taken $f(x)=x$, $g(y)=y$. Then, Eq. (39) has the form

$$c_{01}+2c_{02}y+c_{11}x+c_{21}x^2+2c_{12}xy+3c_{03}y^2=d_{10}+2d_{20}x+d_{11}y+3d_{30}x^2+2d_{21}xy+d_{12}y^2. \tag{40}$$

It is seen that it is necessary to fulfil the following constrains: $c_{01}=d_{10}$, $d_{11}=2c_{02}$, $c_{11}=2d_{20}$, $d_{12}=3c_{03}$, $c_{21}=3d_{30}$ and $d_{21}=c_{12}$. The solution of Eqs. (37)-(38) for the function J is

$$\begin{aligned}J=&\frac{c_{30}}{4}x^4+d_{30}x^3y+\frac{c_{12}}{2}x^2y^2+c_{03}xy^3+\frac{d_{03}}{4}y^4+\frac{c_{20}}{3}x^3+d_{20}x^2y+c_{02}xy^2+\frac{d_{02}}{3}y^3\\&+\frac{c_{10}}{2}x^2+c_{01}xy+\frac{d_{01}}{2}y^2+c_{00}x+d_{00}y.\end{aligned} \tag{41}$$

There are 14 unknown coefficients c_{ij} and d_{ij}.
Now it is necessary to substitute Eq. (41) into Eq. (12) and consequently to Eq. (13) with the potential given by Eq. (10), where $M=3$. Comparing the terms of the same order, system of 21 algebraic equations is obtained.
A general discussion of the problem is very complex. In this paper, solution of simplified problem is presented. Here, it is supposed that all sixth order cross-terms in the potential are zero, i.e. $V_{51}=V_{42}=V_{33}=V_{24}=V_{15}=0$ and all

fifth order terms are zero, i.e. $V_{50} = V_{41} = V_{32} = V_{23} = V_{14} = V_{05} = 0$. These assumptions lead to reduction of free parameters to 7.
The solution is parametrized by independent real parameters $W_{60} = \sqrt{V_{60}}$, $W_{06} = \sqrt{V_{06}}$, V_{40}, V_{04}, V_{30}, V_{03}, $\alpha = c_{01}$. These parameters were chosen to obtain the resulting equations in a simple form.
The resulting potential has the form

$$V(x,y) = {W_{60}}^2 x^6 + {W_{06}}^2 y^6 + V_{40}x^4 + V_{04}y^4 + 2W_{60}\alpha x^3 y + 2W_{06}\alpha x y^3 + V_{30}x^3 + V_{03}y^3 + \left(\frac{{V_{40}}^2}{4{W_{60}}^2} + \alpha^2 - 3W_{60}\right) x^2 + \left(\frac{{V_{04}}^2}{4{W_{06}}^2} + \alpha^2 - 3W_{06}\right) y^2 + \left(\frac{V_{40}}{W_{60}} + \frac{V_{04}}{W_{06}}\right)\alpha xy + \left(\frac{V_{40}V_{30}}{2{W_{60}}^2} + \frac{V_{03}}{W_{06}}\alpha\right) x + \left(\frac{V_{04}V_{03}}{2{W_{06}}^2} + \frac{V_{30}}{W_{60}}\alpha\right) y. \tag{42}$$

The ground state wave function for this potential is

$$\psi_0 = \exp\left(-\frac{W_{60}}{4}x^4 - \frac{W_{06}}{4}y^4 - \frac{V_{40}}{4W_{60}}x^2 - \frac{V_{04}}{4W_{06}}y^2 - \alpha xy - \frac{V_{30}}{2W_{60}}x - \frac{V_{03}}{2W_{06}}y\right). \tag{43}$$

The corresponding energy equals

$$E_0 = \frac{V_{40}}{2W_{60}} + \frac{V_{04}}{2W_{06}} - \frac{{V_{30}}^2}{4{W_{60}}^2} - \frac{{V_{03}}^2}{4{W_{06}}^2}. \tag{44}$$

The wave function (43) has continuous partial derivatives in the entire plane (x, y). If $W_{60} > 0$ a $W_{06} > 0$, function (43) is quadratically integrable.

3.3.2. Two-dimensional sextic Morse potential

To get a potential of the Morse type, we put $f(x) = e^{-x}$, $g(y) = e^{-y}$. Then, to fulfil Eq. (39), it is necessary to take $d_{10} = c_{01} = d_{20} = c_{02} = 0$, $d_{11} = c_{11}$, $d_{12} = 2c_{12}$ and $c_{21} = 2d_{21}$. The solution of Eqs. (37)-(38) for the function J is

$$J = -\frac{c_{30}}{3}e^{-3x} - d_{21}e^{-2x-y} - c_{12}e^{-x-2y} - \frac{d_{03}}{3}e^{-3y} - \frac{c_{20}}{2}e^{-2x} - c_{11}e^{-x-y} - \frac{d_{02}}{2}e^{2y} - c_{10}e^{-x} - d_{01}e^{-y} + c_{00}x + d_{00}y. \tag{45}$$

There are 11 unknown coefficients c_{ij} and d_{ij}.
Now it is necessary to substitute Eq. (45) into Eq. (12) and consequently to Eq. (13) with the potential given by Eq. (10), where $M = 3$. Comparing the terms of the same order, system of 21 algebraic equations is obtained.
A general solution of this problem can be found, however, it is very complex. In this paper, only one of the simplified solutions is presented. Here, it is supposed that all sixth order cross-terms in the potential are zero, i.e. $V_{51} = V_{42} = V_{33} = V_{24} = V_{15} = 0$ and also $V_{40} = V_{04} = 0$. These assumptions caused reduction of free parameters to 7. The solution is parametrized by independent real parameters $W_{60} = \sqrt{V_{60}}$, $W_{06} = \sqrt{V_{06}}$, V_{50}, V_{05}, $\alpha = c_{00}$, $\beta = d_{00}$, $\gamma = c_{11}$.
The resulting potential has the form

$$V = {W_{60}}^2 e^{-6x} + {W_{06}}^2 e^{-6y} + V_{50}e^{-5x} + 2W_{60}\gamma e^{-4x-y} + 2W_{06}\gamma e^{-x-4y} + V_{05}e^{-5y} + \frac{V_{50}}{W_{60}}\gamma e^{-3x-y} + 2\gamma^2 e^{-2x-2y} + \frac{V_{05}}{W_{06}}\gamma e^{-x-3y} + \left[W_{60}(2\alpha+3) - \frac{{V_{50}}^3}{8{W_{60}}^4}\right] e^{-3x} - \frac{{V_{50}}^2}{4{W_{60}}^3}\gamma e^{-2x-y} - \frac{{V_{05}}^2}{4{W_{06}}^3}\gamma e^{-x-2y} + \left[W_{06}(2\beta+3) - \frac{{V_{05}}^3}{8{W_{06}}^4}\right] e^{-3y} + \left[\frac{V_{50}(\alpha+1)}{W_{60}} + \frac{{V_{50}}^4}{64{W_{60}}^6}\right] e^{-2x} + \left[\frac{V_{05}(\beta+1)}{W_{06}} + \frac{{V_{05}}^4}{64{W_{06}}^6}\right] e^{-2y} + 2(\alpha+\beta+1)\gamma e^{-x-y} - \frac{2\alpha+1}{8{W_{60}}^3}{V_{50}}^2 e^{-x} - \frac{2\beta+1}{8{W_{06}}^3}{V_{05}}^2 e^{-y}. \tag{46}$$

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The ground state wave function has the formula

$$\psi_0 = \exp\left(\frac{W_{60}}{3}e^{-3x} + \frac{V_{50}}{4W_{60}}e^{-2x} - \frac{{V_{50}}^2}{8{W_{60}}^3}e^{-x} + \frac{W_{06}}{3}e^{-3y} + \frac{V_{05}}{4W_{06}}e^{-2y} - \frac{{V_{05}}^2}{8{W_{06}}^3}e^{-y} - \alpha x - \beta y + \gamma e^{-x-y}\right) \quad (47)$$

and the corresponding energy equals

$$E_0 = -\alpha^2 - \beta^2. \quad (48)$$

The ground state wave function (47) has continuous derivatives in the entire plane (x, y). If $\alpha > 0$, $\beta > 0$, $W_{60} < 0$ and $W_{06} < 0$, wave function (47) is also quadratically integrable in the entire plane (x, y).

4. Conclusions

Most of known analytic solutions of the two-dimensional and three-dimensional Schrödinger equation, as for instance the quantization of the angular momentum or the hydrogen atom, have been obtained by reducing the problem to one-dimensional differential equations. Problems that cannot be reduced to one-dimensional equations are difficult to solve.

In our preceding papers [9–11], a few analytic solutions of the two-dimensional Schrödinger equation with the quartic polynomial potential were found. The method used in these papers is based on the approach suggested in [6–8]. This method yields most of known analytical solutions of the one-dimensional Schrödinger equation. In general, in one dimension and consequently in two dimensions, analytic solutions for some energies and wave functions are possible only for certain values of the potential coefficients.

In this paper, we used generalized assumptions comprising wider class of two-dimensional potentials (Sections 2 and 3). Then, we applied this method to the fourth and sixth order two-dimensional polynomial and Morse potentials. In these cases, the analytic ground state energies and wave functions for some values of the potential coefficients were found (Section 3). The wave functions have continuous partial derivatives and are quadratically integrable in the entire plane (x, y).

These examples show that despite complexity of the problem, analytic solutions of the two-dimensional Schrödinger equation can be found in some physically interesting cases that cannot be reduced to one-dimensional problems.

For the one-dimensional problems, no analytic methods for calculating the complete energy spectrum of the Schrödinger equation with the fourth and higher order potentials are known. In two dimensions, the situation is even worse and our contribution to this effort is summarized in this paper. We have shown that the analytic ground state energies and wave functions can be found in many cases discussed in this paper. Solutions for the excited states are even more difficult and have been found in only one case [9]. Other excited states will be the subject of further research.

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