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Analytic wave functions and energies for two-dimensional $\mathcal{PT}$-symmetric quartic potentials

Research Article

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Abstract: Analytic wave functions and the corresponding energies for a class of the $\mathcal{PT}$-symmetric two-dimensional quartic potentials are found. The general form of the solutions is discussed.

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1. Introduction

In this paper, we are interested in the solutions of the two-dimensional Schrödinger equation with the quartic $\mathcal{PT}$-symmetric potential $V(x, y)$

$$-\Delta\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y), \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2}$$

Here, the potential V is assumed to be in the form

$$\begin{aligned} V(x,y) &= V_{40}x^4 + V_{04}y^4 + V_{31}x^3y + V_{13}xy^3 \\ &+ V_{22}x^2y^2 + \mathrm{i}V_{30}x^3 + \mathrm{i}V_{03}y^3 + \mathrm{i}V_{21}x^2y + \mathrm{i}V_{12}xy^2 \\ &+ V_{20}x^2 + V_{02}y^2 + V_{11}xy + \mathrm{i}V_{10}x + \mathrm{i}V_{01}y, \end{aligned} \tag{3}$$

where the V_{ij} are real coefficients. We suppose $V_{40} > 0$ and $V_{04} > 0$. For more information on the $\mathcal{PT}$-symmetric potentials, see e.g. [1–7].

To find analytic solutions, a similar approach as in [8–11] is used. Here, the ground state wave function is searched for in the form of the exponential of a cubic polynomial in the variables x and y

$$\psi(x,y) = \exp\left[-P(x,y)\right], \tag{4}$$

where

$$\begin{aligned} P(x,y) &= c_{30}x^3 + c_{03}y^3 + c_{21}x^2y + c_{12}xy^2 \\ &+ c_{20}x^2 + c_{02}y^2 + c_{11}xy + c_{10}x + c_{01}y. \end{aligned} \tag{5}$$

Here, the c_{ij} are complex coefficients to be found. The normalization factor is not written here. Note that in the papers [8–11], excited states of polynomial potentials are searched in the form of an exponential of a polynomial

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multiplied by another polynomial. In this paper, only the ground state is presented.
Eq. (1) can be rewritten in the form

$$V(x,y) - E = \frac{\Delta\psi(x,y)}{\psi(x,y)}. \tag{6}$$

Substituting Eqs. (3)-(5) into Eq. (6) we get an equation with polynomials in both sides. Comparing the terms of the same order we get the formula for the energy

$$E = -{c_{10}}^2 - {c_{01}}^2 + 2c_{02} + 2c_{20} \tag{7}$$

and the system of equations for the coefficients c_{ij}

$$\begin{aligned} V_{40} &= 9{c_{30}}^2 + {c_{21}}^2, && (8)\\ V_{04} &= 9{c_{03}}^2 + {c_{12}}^2, && (9)\\ V_{31} &= 12c_{30}c_{21} + 4c_{21}c_{12}, && (10)\\ V_{13} &= 12c_{03}c_{12} + 4c_{12}c_{21}, && (11)\\ V_{22} &= 6c_{30}c_{12} + 4{c_{12}}^2 + 4{c_{21}}^2 + 6c_{21}c_{03}, && (12) \end{aligned}$$

$$\begin{aligned} \mathrm{i}V_{30} &= 12c_{20}c_{30} + 2c_{11}c_{21}, && (13)\\ \mathrm{i}V_{03} &= 12c_{02}c_{03} + 2c_{11}c_{12}, && (14)\\ \mathrm{i}V_{21} &= 4c_{21}c_{02} + 8c_{20}c_{21} + 6c_{30}c_{11} + 4c_{11}c_{12}, && (15)\\ \mathrm{i}V_{12} &= 4c_{12}c_{20} + 8c_{02}c_{12} + 6c_{03}c_{11} + 4c_{11}c_{21}, && (16) \end{aligned}$$

$$\begin{aligned} V_{20} &= 2c_{01}c_{21} + {c_{11}}^2 + 4{c_{20}}^2 + 6c_{10}c_{30}, && (17)\\ V_{02} &= 2c_{10}c_{12} + {c_{11}}^2 + 4{c_{02}}^2 + 6c_{01}c_{03}, && (18)\\ V_{11} &= 4c_{01}c_{12} + 4c_{11}c_{02} + 4c_{20}c_{11} + 4c_{10}c_{21}, && (19)\\ \mathrm{i}V_{10} &= 4c_{10}c_{20} - 6c_{30} + 2c_{01}c_{11} - 2c_{12}, && (20)\\ \mathrm{i}V_{01} &= 4c_{01}c_{02} - 6c_{03} + 2c_{10}c_{11} - 2c_{21}. && (21) \end{aligned}$$

This is a system of 14 equations for 9 coefficients. It is evident that this system is not solvable in general and certain conditions of its solvability must be fulfilled.

2. Energy and conditions of solvability

To get the $\mathcal{PT}$-symmetric solutions we suppose that all coefficients c_{ij} with $i+j$ even are purely imaginary and all c_{ij} coefficients with $i+j$ odd are real. $\mathcal{PT}$-symmetry will be supposed to be unbroken and for this reason the energy has to be real. From Eq. (7), it follows that

$$c_{02} = -c_{20}. \tag{22}$$

The left hand side of Eqs. (20) and (21) is purely imaginary, and thus the right hand side must be purely imaginary too. From here, it follows that

$$\begin{aligned} c_{21} &= -3c_{03}, && (23)\\ c_{12} &= -3c_{30}. && (24) \end{aligned}$$

The system of Eqs. (8)-(21) then becomes

$$\begin{aligned} V_{40} &= 9{c_{30}}^2 + 9{c_{03}}^2, && (25)\\ V_{04} &= 9{c_{30}}^2 + 9{c_{03}}^2, && (26)\\ V_{31} &= 0, && (27)\\ V_{13} &= 0, && (28)\\ V_{22} &= 18{c_{30}}^2 + 18{c_{03}}^2, && (29) \end{aligned}$$

$$\begin{aligned} \mathrm{i}V_{30} &= 12c_{30}c_{20} - 6c_{03}c_{11}, && (30)\\ \mathrm{i}V_{03} &= -12c_{03}c_{20} - 6c_{30}c_{11}, && (31)\\ \mathrm{i}V_{21} &= -12c_{03}c_{20} - 6c_{30}c_{11}, && (32)\\ \mathrm{i}V_{12} &= 12c_{30}c_{20} - 6c_{03}c_{11}, && (33) \end{aligned}$$

$$\begin{aligned} V_{20} &= 6c_{30}c_{10} - 6c_{03}c_{01} + 4{c_{20}}^2 + {c_{11}}^2, && (34)\\ V_{02} &= -6c_{30}c_{10} + 6c_{03}c_{01} + 4{c_{20}}^2 + {c_{11}}^2, && (35)\\ V_{11} &= -12c_{30}c_{01} - 12c_{03}c_{10}, && (36)\\ \mathrm{i}V_{10} &= 4c_{20}c_{10} + 2c_{11}c_{01}, && (37)\\ \mathrm{i}V_{01} &= -4c_{20}c_{01} + 2c_{11}c_{10}. && (38) \end{aligned}$$

Immediately, one can see the following conditions of solvability: Eqs. (27), (28) give $V_{31} = 0$ and $V_{13} = 0$. Eqs. (25), (26), (29) yield further conditions

$$V_{40} = V_{04} = \frac{V_{22}}{2} \equiv V_4. \tag{39}$$

From Eqs. (30) and (33), resp. (31) and (32) follow the conditions

$$\begin{aligned} V_{12} &= V_{30}, && (40)\\ V_{21} &= V_{03}. && (41) \end{aligned}$$

Now, we can express from Eqs. (30) and (31) the coefficients

$$\begin{aligned} c_{20} &= \frac{1}{12}\frac{V_{30}c_{30} - V_{03}c_{03}}{{c_{30}}^2 + {c_{03}}^2}\mathrm{i}, && (42)\\ c_{11} &= \frac{1}{6}\frac{V_{30}c_{03} + V_{03}c_{30}}{{c_{30}}^2 + {c_{03}}^2}\mathrm{i}. && (43) \end{aligned}$$

Substituting these coefficients into Eqs. (37) and (38) and solving the resulting two equations one gets

$$c_{10} = 3\frac{V_{30}V_{10} - V_{03}V_{01}}{{V_{30}}^2 + {V_{03}}^2}c_{30} - 3\frac{V_{03}V_{10} + V_{30}V_{01}}{{V_{30}}^2 + {V_{03}}^2}c_{03}, \quad (44)$$

$$c_{01} = -3\frac{V_{03}V_{10} + V_{30}V_{01}}{{V_{30}}^2 + {V_{03}}^2}c_{30} + 3\frac{-V_{30}V_{10} + V_{03}V_{01}}{{V_{30}}^2 + {V_{03}}^2}c_{03}. \quad (45)$$

Now, it is useful to express the sum of Eqs. (34) and (35) and their difference:

$$V_{20} + V_{02} = 8{c_{20}}^2 + 2{c_{11}}^2, \quad (46)$$

$$V_{20} - V_{02} = 12c_{10}c_{30} - 12c_{03}c_{01}. \quad (47)$$

Substituting Eqs. (42)-(45) to Eqs. (36), (46) and (47) one obtains

$$V_{11} = 36\frac{\left({c_{30}}^2 + {c_{03}}^2\right)\left(V_{30}V_{01} + V_{03}V_{10}\right)}{{V_{30}}^2 + {V_{03}}^2}, \quad (48)$$

$$V_{20} + V_{02} = \frac{1}{18}\frac{{V_{30}}^2 + {V_{03}}^2}{{c_{30}}^2 + {c_{03}}^2}, \quad (49)$$

$$V_{20} - V_{02} = 36\frac{\left({c_{30}}^2 + {c_{03}}^2\right)\left(V_{30}V_{10} - V_{03}V_{01}\right)}{{V_{30}}^2 + {V_{03}}^2}. \quad (50)$$

Referring to Eq. (25) we can rewrite Eqs. (48)-(50) in the form

$$V_{11} = 4\frac{V_4(V_{30}V_{01} + V_{03}V_{10})}{{V_{30}}^2 + {V_{03}}^2}, \quad (51)$$

$$V_{20} + V_{02} = \frac{-1}{2}\frac{{V_{30}}^2 + {V_{03}}^2}{V_4}, \quad (52)$$

$$V_{20} - V_{02} = 4\frac{V_4(V_{30}V_{10} - V_{03}V_{01})}{{V_{30}}^2 + {V_{03}}^2}. \quad (53)$$

These equations are the remaining three conditions of solvability. Denoting

$$V_a \equiv V_{30}V_{10} - V_{03}V_{01}, \quad (54)$$

$$V_b \equiv V_{30}V_{01} + V_{03}V_{10}. \quad (55)$$

Eq. (3) describing the potential can be rewritten to the form

$$V(x,y) = V_4x^4 + V_4y^4 + 2V_4x^2y^2 + \mathrm{i}V_{30}x^3 + \mathrm{i}V_{03}y^3 + \mathrm{i}V_{03}x^2y + \mathrm{i}V_{30}xy^2 + \left(\frac{-1}{4}\frac{{V_{30}}^2 + {V_{03}}^2}{V_4} + \frac{2V_4V_a}{{V_{30}}^2 + {V_{03}}^2}\right)x^2$$
$$+ \left(\frac{-1}{4}\frac{{V_{30}}^2 + {V_{03}}^2}{V_4} - \frac{2V_4V_a}{{V_{30}}^2 + {V_{03}}^2}\right)y^2 + \frac{4V_4V_b}{{V_{30}}^2 + {V_{03}}^2}xy + \mathrm{i}\frac{V_{30}V_a + V_{03}V_b}{{V_{30}}^2 + {V_{03}}^2}x + \mathrm{i}\frac{V_{30}V_b - V_{03}V_a}{{V_{30}}^2 + {V_{03}}^2}y. \quad (56)$$

Eq. (56) represents the general form of the resulting potential fulfilling all mentioned conditions. Here, V_4 is an arbitrary positive real number and V_{30}, V_{03}, V_a and V_b are arbitrary real numbers.
Transformation of Eq. (56) to spherical coordinates

$$x = R\cos(\phi), \quad (57)$$

$$y = R\sin(\phi), \quad (58)$$

gives

$$V(R,\phi) = V_4R^4 + \mathrm{i}V_{30}R^3\cos(\phi) + \mathrm{i}V_{03}R^3\sin(\phi) + \left(\frac{-1}{4}\frac{{V_{30}}^2 + {V_{03}}^2}{V_4} + \frac{2V_4V_a}{{V_{30}}^2 + {V_{03}}^2}\right)R^2\cos^2(\phi)$$
$$+ \left(\frac{-1}{4}\frac{{V_{30}}^2 + {V_{03}}^2}{V_4} - \frac{2V_4V_a}{{V_{30}}^2 + {V_{03}}^2}\right)R^2\sin^2(\phi) + \frac{4V_4V_b}{{V_{30}}^2 + {V_{03}}^2}R^2\cos(\phi)\sin(\phi)$$
$$+ \mathrm{i}\frac{V_{30}V_a + V_{03}V_b}{{V_{30}}^2 + {V_{03}}^2}R\cos(\phi) + \mathrm{i}\frac{V_{30}V_b - V_{03}V_a}{{V_{30}}^2 + {V_{03}}^2}R\sin(\phi). \quad (59)$$

Eq. (59) shows a non-trivial dependency of the potential on the angular coordinate ϕ. It is seen that the presented method leads to a new type of $\mathcal{PT}$-symmetric potential.
Substituting Eqs. (42)-(45) into the energy (7) we obtain

$$E = -9\frac{\left({c_{30}}^2 + {c_{03}}^2\right)\left({V_{10}}^2 + {V_{01}}^2\right)}{{V_{30}}^2 + {V_{03}}^2}. \tag{60}$$

Using Eq. (25) and (54)-(55) this formula becomes

$$E = -V_4\frac{{V_a}^2 + {V_b}^2}{\left({V_{30}}^2 + {V_{03}}^2\right)^2}. \tag{61}$$

3. General form of the wave function

Using Eq. (25), Eqs. (42)-(43) can be rewritten in the form

$$c_{20} = \frac{3}{4}\frac{V_{30}c_{30} - V_{03}c_{03}}{V_{40}}\mathrm{i}, \tag{62}$$

$$c_{11} = -\frac{3}{2}\frac{V_{30}c_{03} + V_{03}c_{30}}{V_{40}}\mathrm{i}. \tag{63}$$

The general solution of Eqs. (25)-(29) can be written as

$$c_{30} = \frac{\sqrt{V_4}}{3}\cos(\alpha), \tag{64}$$

$$c_{03} = \frac{\sqrt{V_4}}{3}\sin(\alpha), \tag{65}$$

where α is a real parameter within the interval $[0, 2\pi)$. It will be seen that α can be an arbitrary real number within this interval.
Now it is possible to write Eq. (5) in the form

$$P(x,y) = \frac{\sqrt{V_4}}{3}\cos(\alpha)P_1(x,y) + \frac{\sqrt{V_4}}{3}\sin(\alpha)P_2(x,y), \tag{66}$$

where

$$P_1(x,y) = x^3 - 3xy^2 + \mathrm{i}\frac{3V_{30}}{4V_4}x^2 - \mathrm{i}\frac{3V_{30}}{4V_4}y^2 - \mathrm{i}\frac{3V_{03}}{2V_4}xy + \frac{3V_a}{{V_{30}}^2 + {V_{03}}^2}x - \frac{3V_b}{{V_{30}}^2 + {V_{03}}^2}y, \tag{67}$$

$$P_2(x,y) = y^3 - 3x^2y - \mathrm{i}\frac{3V_{03}}{4V_4}x^2 + \mathrm{i}\frac{3V_{03}}{4V_4}y^2 - \mathrm{i}\frac{3V_{30}}{2V_4}xy - \frac{3V_b}{{V_{30}}^2 + {V_{03}}^2}x - \frac{3V_a}{{V_{30}}^2 + {V_{03}}^2}y. \tag{68}$$

The resulting wave function for the potential of the form (3) fulfilling the conditions (27), (28), (39)-(41) and (51)-(53) can be written as

$$\psi(x,y) = \exp\left[-\frac{\sqrt{V_4}}{3}\left[\cos(\alpha)P_1(x,y) + \sin(\alpha)P_2(x,y)\right]\right], \tag{69}$$

where $P_1(x,y)$ and $P_2(x,y)$ are defined by Eqs. (67)-(68). The corresponding energy is given by Eq. (61). The parameter α is a real number in the interval $[0, 2\pi)$. In general, the wave functions (69) corresponding to different values of α are linearly independent. The energy (61) does not depend on α. For this reason, this energy is infinitely degenerate.

4. Conclusions

In this paper, the two-dimensional Schrödinger equation with the $\mathcal{PT}$-symmetric quartic potential has been investigated. The analytic formulae for ground state energy and the corresponding wave functions have been found. This energy is infinitely degenerate. Further investigation of this problem will be the subject of further research. Also, other solutions will be searched.

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References

[1] F. Cannata, G. Junker, J. Trost, Phys. Lett. A 246, 219 (1998)
[2] M. Znojil, J. Phys. A 36, 7825 (2003)
[3] M. Znojil, J. Phys. A 35, 2341 (2002)
[4] M. Znojil, J. Phys. A 36, 7639 (2003)
[5] E. Delabaere, F. Pham, Phys. Lett. A 250, 25 (1998)
[6] E. Delabaere, F. Pham, Phys. Lett. A 250, 29 (1998)
[7] G. Lévai, J. Phys. A 40, F273 (2007)
[8] V. Tichý, L. Skála, Collect. Czech. Chem. C. 73, 1327 (2008)
[9] L. Skála, J. Čížek, J. Dvořák, V. Špirko, Phys. Rev. A 53, 2009 (1996)
[10] J. Dvořák, L. Skála, Collect. Czech. Chem. C. 63, 1161 (1998)
[11] L. Skála, J. Dvořák, V. Kapsa, Int. J. Theor. Phys. 36, 2953 (1997)