Analytic energies and wave functions of the two-dimensional Schrödinger equation: ground state of two-dimensional quartic potential and classification of solutions

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Abstract: New analytic solutions of the two-dimensional Schrödinger equation with a two-dimensional fourth-order polynomial (i.e., quartic) potential are derived and discussed. The solutions represent the ground state energies and the corresponding wave functions. In general, the obtained results cannot be reduced to two one-dimensional cases.

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Résumé : Nous obtenons et analysons de nouvelles solutions analytiques de l'équation de Schrödinger en 2-D avec un potentiel polynomial quartique en 2-D. Les solutions représentent les énergies du fondamental et les fonctions d'onde correspondantes. En général, les résultats ne peuvent pas être réduits à deux cas unidimensionnels.

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1. Introduction

The Schrödinger equation represents the fundamental equation of quantum mechanics. This paper is concerned with its time-independent form in two dimensions

$$-\Delta\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y) \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The function $V(x, y)$ is the potential (i.e., a given real function representing physical problem). In this paper, potential V is assumed in the form of the fourth-order polynomial:

$$V(x,y) = \sum_{\substack{m\geq 0\, n\geq 0 \\ m+n\leq 4}} V_{mn}x^m y^n \tag{2}$$

where V_{mn} are real constants. Potential (2) represents a wide class of physical problems including quantum anharmonic oscillator [1, 2] and quantum double-well problems. These models are widely used, for example, in chemical physics.

Equation (1) represents a partial differential equation. When we speak about solving (1), we mean two problems. The first problem is to find such values of E that (1) has a solution, that is, we search for the eigenvalues of the operator $-\Delta + V(x, y)$. The second problem is to find the corresponding complex functions $\psi(x, y)$ of real variables x and y solving (1). In this paper, we restrict ourselves to searching for the ground states (i.e., for the lowest value of E, denoted E_0, and for the corresponding wave function ψ, denoted ψ_0). We search for the analytical solutions, that is, for their formulae in the closed form.

Commonly the solutions $\psi(x, y)$ are required to be quadratically integrable in the whole plane (x, y), that is, the boundary condition

$$\int_{\mathbf{R}^2} \psi(x,y)\psi^*(x,y)\mathrm{d}x\mathrm{d}y < +\infty \tag{3}$$

has to be fulfilled. The asterisk denotes complex conjugation.

The common method to find solutions of (1) is to separate it into two ordinary (i.e., one-dimensional) differential equations [3–5]. To solve (1) when it is impossible to separate it into two ordinary differential equations, the proper methods are called for.

The method presented here is based on the method for one-dimensional problems, given in refs. 6–8. There, it is assumed that solutions $\psi(x)$ of the one-dimensional Schrödinger equation

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$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \tag{4}$$

are linear combinations of the functions $\psi_m(x)$ in the form

$$\psi(x) = \sum_m \alpha_m \psi_m(x) \tag{5}$$

where

$$\psi_m(x) = f^m(x)h(x) \tag{6}$$

It has been proved that to obtain analytic solutions, the potential V must have the form

$$V(x) = \sum_m V_m f(x)^m \tag{7}$$

Function $h(x)$ is expected to appear in the form [1, 2]

$$h(x) = \exp\left(-\int \sum_m h_m f^m(x)\,dx\right) \tag{8}$$

Using this approach, it is possible to take different forms of $f(x)$ and to test if the Schrödinger equation with potential V of the form (7) has analytical solutions obeying the corresponding boundary condition. In (7), the limits for the index m are given by studied potential and if necessary, negative values of m can also be admitted. After substituting (5)–(8) into (4), a system of algebraic equations for unknowns d_m and h_m is obtained. Allowed values of indices m in (5) and (8) follow from the condition that the number of obtained equations has to be equal to the number of unknowns.

In ref. 9, the first attempt was made to generalize this method to two dimensions. There, the Schrödinger equation has the form (1). Its solutions $\psi(x, y)$ are assumed in the form

$$\psi(x, y) = \sum_{m,n} c_{mn}\psi_{mn} \tag{9}$$

where

$$\psi_{mn}(x, y) = f^m(x)g^n(y)h(x, y) \tag{10}$$

The potential $V(x, y)$ is assumed in the form

$$V(x, y) = \sum_{m,n} V_{mn} f^m(x)g^n(y) \tag{11}$$

For the polynomial potential, we take $f(x) = x$ and $g(y) = y$. Here, we aim to the fourth-order polynomial potential, so the sum in (11) is performed over all $m \in \{0, 1, 2, 3, 4\}$ and $n \in \{0, 1, 2, 3, 4\}$ excluding cases $m + n > 4$. Term V_{00} represents an irrelevant additive factor, so that we assume $V_{00} = 0$. We get

$$\begin{aligned} V(x, y) = {} & W_{40}^2x^4 + W_{04}^2y^4 + V_{31}x^3y + V_{13}xy^3 + V_{22}x^2y^2 \\ & + V_{30}x^3 + V_{03}y^3 + V_{21}x^2y + V_{12}xy^2 + V_{20}x^2 \\ & + V_{02}y^2 + V_{11}xy + V_{10}x + V_{01}y \end{aligned} \tag{12}$$

where $W_{40} = \pm\sqrt{V_{40}}$ and $W_{04} = \pm\sqrt{V_{04}}$. The sign of the coefficients W_{40} and W_{04} is discussed below. We assume that $V_{40} > 0$ and $V_{04} > 0$, which are necessary conditions for the existence of solutions fulfilling boundary condition (3).

Generalization of (8) to two dimensions has not been found. However, it has been shown in ref. 4 that for the two-dimensional quartic polynomial, the function h is equal to the ground state wave function ψ_0 and it has the form

$$h(x, y) = \psi_0(x, y) = \exp\left(\sum_{\substack{i\geq 0\, j\geq 0 \\ i+j\leq 3}} - d_{ij}x^iy^j\right) \tag{13}$$

where the coefficients d_{ij} have to be found. One of the existing solutions is presented in ref. 9. Searching for general formulae for those coefficients and consequently for formulae for ground state wave functions and the corresponding energies is the subject of this paper.

Wave function (13) is not quadratically integrable in the whole plane (x, y). One possible approach to solving this problem is to suppose that we solve the Schrödinger equation on the quadrant $x \geq 0$, $y \geq 0$. In this paper another approach was chosen. The wave function was modified in the same way as in refs. 6–9 to

$$\begin{aligned} \psi_0(x, y) = \exp\big(& -d_{30}|x|^3 - d_{03}|y|^3 - d_{21}x^2|y| - d_{12}|x|y^2 \\ & - d_{20}x^2 - d_{02}y^2 - d_{11}|xy| - d_{10}|x| - d_{01}|y|\big) \end{aligned} \tag{14}$$

This function solves the Schrödinger equation with the modified potential depending on $|x|$ and $|y|$

$$\begin{aligned} V(x, y) = {} & W_{40}^2x^4 + W_{04}^2y^4 + V_{31}|x|^3y + V_{13}x|y|^3 \\ & + V_{22}x^2y^2 + V_{30}|x|^3 + V_{03}|y|^3 + V_{21}x^2|y| + V_{12}|x|y^2 \\ & + V_{20}x^2 + V_{02}y^2 + V_{11}|xy| + V_{10}|x| + V_{01}|y| \end{aligned} \tag{15}$$

This approach has a disadvantage in that the wave function (14) and the potential (15) do not have continuous derivatives on axes $x = 0$ and $y = 0$.

Substituting (12) and (13) into (1) and comparing the terms of equal order yields the formula for the ground state energy [4]

$$E_0 = 2d_{20} + 2d_{02} - d_{10}^2 - d_{01}^2 \tag{16}$$

The system of equations for the wave function coefficients d_{ij} becomes

$$W_{40}^2 = 9d_{30}^2 + d_{21}^2 \tag{17}$$

$$W_{04}^2 = 9d_{03}^2 + d_{12}^2 \tag{18}$$

$$V_{31} = 12d_{30}d_{21} + 4d_{21}d_{12} \tag{19}$$

$$V_{13} = 12d_{03}d_{12} + 4d_{12}d_{21} \tag{20}$$

$$V_{22} = 6d_{30}d_{12} + 4d_{12}^2 + 4d_{21}^2 + 6d_{21}d_{03} \tag{21}$$

$$V_{30} = 12d_{20}d_{30} + 2d_{11}d_{21} \tag{22}$$

$$V_{03} = 12d_{02}d_{03} + 2d_{11}d_{12} \quad (23)$$

$$V_{21} = 4d_{21}d_{02} + 8d_{20}d_{21} + 6d_{30}d_{11} + 4d_{11}d_{12} \quad (24)$$

$$V_{12} = 4d_{12}d_{20} + 8d_{02}d_{12} + 6d_{03}d_{11} + 4d_{11}d_{21} \quad (25)$$

$$V_{20} = 2d_{01}d_{21} + d_{11}^2 + 4d_{20}^2 + 6d_{10}d_{30} \quad (26)$$

$$V_{02} = 2d_{10}d_{12} + d_{11}^2 + 4d_{02}^2 + 6d_{01}d_{03} \quad (27)$$

$$V_{11} = 4d_{01}d_{12} + 4d_{11}d_{02} + 4d_{20}d_{11} + 4d_{10}d_{21} \quad (28)$$

$$V_{10} = 4d_{10}d_{20} - 6d_{30} + 2d_{01}d_{11} - 2d_{12} \quad (29)$$

$$V_{01} = 4d_{01}d_{02} - 6d_{03} + 2d_{10}d_{11} - 2d_{21} \quad (30)$$

This is a system of 14 equations for nine coefficients d_{ij}. It is evident that this system is not solvable in general. However, for certain choices of the potential coefficients V_{ij} some of the preceding equations become dependent and a regular system of equations is obtained. This problem is well known from the one-dimensional case, where only some quartic potentials are analytically solvable [6–8].

2. Classification of the solutions

We prove in Appendix A that each two-dimensional quartic polynomial can be transformed into a polynomial with $V_{31} = 0$. Therefore, without any loss of generality we can assumme $V_{31} = 0$.

Table 1. Labeling of solutions.

	$d_{12} = 0$, $d_{21} = 0$	$d_{12} = 0$, $d_{21} \neq 0$	$d_{12} \neq 0$, $d_{21} = 0$	$d_{12} \neq 0$, $d_{21} \neq 0$
$V_{13} = 0$, $V_{31} = 0$	aI	bI	cI	dI
$V_{13} \neq 0$, $V_{31} = 0$	aII	bII	cII	dII

In (19) and (20) it is necessary to discuss whether the values of the variables V_{13}, d_{21}, and d_{12} are zero or nonzero. We denote the corresponding solutions as “a type” for $d_{12} = 0$ and $d_{21} = 0$, “b type” for $d_{12} = 0$ and $d_{21} \neq 0$, “c type” for $d_{12} \neq 0$ and $d_{21} = 0$, and “d type” for $d_{12} \neq 0$ and $d_{21} \neq 0$. Further, we denote solutions as type I for $V_{13} = 0$ and II for $V_{13} \neq 0$. In total, we get eight solution classes: aI, bI, cI, dI, aII, bII, cII, and dII listed in Table 1.

It is seen from (17)–(30) that only some cases have to be considered, because bI and cI classes are equivalent and one of them can be obtained from the other by interchanging $d_{ij} \leftrightarrow d_{ji}$, $W_{ij} \leftrightarrow W_{ji}$, and $V_{ij} \leftrightarrow V_{ji}$. Further, classes aII and bII are empty, because if $V_{13} \neq 0$ and $d_{12} = 0$ then (20) has no solution.

2.1. Solutions of type aI

In this case, we suppose $V_{13} = V_{31} = d_{21} = d_{12} = 0$. This case is solved in ref. 9 using the parameter

$$\alpha \equiv \frac{V_{21}}{W_{40}} = \frac{V_{12}}{W_{04}} \quad (31)$$

The resulting potential has the form

$$V_0(x,y) = W_{40}^2x^4 + W_{04}^2y^4 + V_{30}|x|^3 + V_{03}|y|^3 + W_{40}\alpha x^2|y| + W_{04}\alpha|x|y^2 + V_{20}x^2 + V_{02}y^2 + \frac{\alpha}{2}\left(\frac{V_{30}}{W_{40}} + \frac{V_{03}}{W_{04}}\right)|xy| + \left(\frac{4W_{04}^2V_{02} - W_{40}^2\alpha^2 - V_{03}}{8W_{04}^3}\alpha + \frac{4W_{40}^2V_{20} - W_{40}^2\alpha^2 - V_{30}^2}{8W_{40}^4}V_{30} - 2W_{40}\right)|x| + \left(\frac{4W_{40}^2V_{20} - W_{40}^2\alpha^2 - V_{30}^2}{8W_{40}^3}\alpha + \frac{4W_{04}^2V_{02} - W_{04}^2\alpha^2 - V_{03}^2}{8W_{04}^4}V_{03} - 2W_{04}\right)|y| \quad (32)$$

where W_{40} and W_{04} are arbitrary positive real numbers and V_{30}, V_{03}, V_{20}, V_{02}, and α are arbitrary real numbers. The ground state wave function for potential (32) has the form

$$\psi_0(x,y) = \exp\left(-\frac{W_{40}}{3}|x|^3 - \frac{W_{04}}{3}|y|^3 - \frac{V_{30}}{4W_{40}}x^2 - \frac{V_{03}}{4W_{04}}y^2 - \frac{\alpha}{2}|xy| - \frac{4W_{40}^2V_{20} - W_{40}^2\alpha^2 - V_{30}^2}{8W_{40}^3}|x| - \frac{4W_{40}^2V_{02} - W_{40}^2\alpha^2 - V_{03}^2}{8W_{40}^3}|y|\right) \quad (33)$$

The corresponding ground state energy is

$$E_0 = \frac{V_{30}}{2W_{40}} + \frac{V_{03}}{2W_{04}} - \frac{\left(4W_{40}^2V_{20} - W_{40}^2\alpha^2 - V_{30}^2\right)^2}{64W_{40}^6} - \frac{\left(4W_{04}^2V_{02} - W_{04}^2\alpha^2 - V_{03}^2\right)^2}{64W_{04}^6} \quad (34)$$

Resulting wave function (33) is quadratically integrable in the whole plane (x, y), because W_{40} and W_{04} are supposed to be positive.

2.2. Solutions of type cI

Here, we suppose that $V_{13} = V_{31} = d_{21} = 0$ and $d_{12} \neq 0$. Equation (19) is fulfilled automatically and (17), (18), (20), and (21) have the form

$$W_{40}^2 = 9d_{30}^2 \quad (35)$$

$$W_{04}^2 = 9d_{03}^2 + d_{12}^2 \quad (36)$$

$$0 = 12d_{03}d_{12} \quad (37)$$

$$V_{22} = 6d_{30}d_{12} + 4d_{12}^2 \quad (38)$$

The solution and condition of the solvability of this system are

$$d_{30} = \frac{W_{40}}{3} \quad (39)$$

$$d_{03} = 0 \quad (40)$$

$$d_{12} = W_{04} \quad (41)$$

$$V_{22} = 2W_{40}W_{04} + 4W_{04}^2 \quad (42)$$

Substituting (39)–(41) into (22)–(25) we get

$$V_{30} = 4d_{20}W_{40} \quad (43)$$

$$V_{03} = 2d_{11}W_{04} \quad (44)$$

$$V_{21} = 4d_{11}W_{04} + 2W_{40}d_{11} \quad (45)$$

$$V_{12} = 4d_{20}W_{04} + 8d_{02}W_{04} \quad (46)$$

The solution of this system of equations and the condition of its solvability are

$$d_{20} = \frac{V_{30}}{4W_{40}} \quad (47)$$

$$d_{02} = \frac{V_{12}}{8W_{04}} - \frac{V_{30}}{8W_{40}} \quad (48)$$

$$d_{11} = \frac{V_{03}}{2W_{04}} \quad (49)$$

$$V_{21} = V_{03}\left(\frac{W_{40}}{W_{04}} + 2\right) \quad (50)$$

Substituting (39)–(41) and (47)–(49) into (26)–(30) we get

$$V_{20} = \frac{V_{30}^2}{4W_{40}^2} + 2d_{10}W_{40} + \frac{V_{03}^2}{4W_{04}^2} \quad (51)$$

$$V_{02} = \frac{(V_{12}W_{40} - V_{30}W_{04})^2}{16W_{40}^2W_{04}^2} + 2d_{10}W_{04} + \frac{V_{03}^2}{4W_{04}^2} \quad (52)$$

$$V_{11} = \frac{V_{30}V_{03}}{2W_{04}W_{40}} + 4d_{01}W_{04} + \frac{V_{03}(V_{12}W_{40} - V_{30}W_{04})}{4W_{04}^2W_{40}} \quad (53)$$

$$V_{10} = \frac{d_{01}V_{03}}{W_{04}} + \frac{d_{10}V_{30}}{W_{40}} - 2W_{04} - 2W_{40} \quad (54)$$

$$V_{01} = \frac{d_{10}V_{03}}{W_{04}} + \frac{d_{01}(V_{12}W_{40} - V_{30}W_{04})}{2W_{04}W_{40}} \quad (55)$$

Coefficients d_{10} and d_{01} can be expressed from (51) and (53) as

$$d_{10} = -\frac{V_{30}^2}{8W_{40}^3} - \frac{V_{03}^2}{8W_{40}W_{04}^2} + \frac{V_{20}}{2W_{40}} \quad (56)$$

$$d_{01} = -\frac{V_{30}V_{03}}{16W_{40}W_{04}^2} - \frac{V_{03}V_{12}}{16W_{04}^3} + \frac{V_{11}}{4W_{04}} \quad (57)$$

Equations (52), (54), and (55) give the last conditions of solvability as

$$16W_{40}V_{02} - 16W_{04}V_{20} = \frac{W_{40}(4V_{03}^2 + V_{12})}{W_{04}^2} - \frac{4W_{04}V_{30}^2}{W_{40}^2} + \frac{V_{30}^2}{W_{40}} - \frac{2V_{30}V_{12} + 4V_{03}^2}{W_{04}} \quad (58)$$

and

$$V_{10} = \frac{V_{30}V_{20}}{2W_{40}^2} - \frac{V_{30}^2}{8W_{40}^4} - \frac{V_{30}V_{03}^2}{8W_{40}^2W_{04}^2} - \frac{V_{03}^2V_{12}}{16W_{04}^4} + \frac{V_{03}V_{11}}{4W_{04}^2} - \frac{V_{30}V_{03}^2}{16W_{40}W_{04}^3} - 2W_{04} - 2W_{40} \quad (59)$$

$$V_{01} = \frac{4V_{03}V_{20} - V_{11}V_{30}}{8W_{40}W_{04}} - \frac{V_{30}^2V_{03}}{8W_{40}^3W_{04}} - \frac{V_{03}^3}{8W_{40}W_{04}^3} + \frac{V_{12}V_{11}}{8W_{04}^2} + \frac{V_{30}^2V_{03}}{32W_{40}^2W_{04}^2} - \frac{V_{03}V_{12}^2}{32W_{04}^4} \quad (60)$$

Substituting (39)–(42), (47)–(50), (56), (57), (59), and (60) into (14) we get the following formula for the potential:

$$V(x, y) = W_{40}^2 x^4 + W_{04}^2 y^4 + 2(W_{40}W_{04} + 2W_{04}^2)x^2y^2 + V_{30}|x|^3 + V_{03}|y|^3 + V_{03}\left(\frac{W_{40}}{W_{04}} + 2\right)x^2|y| + V_{12}|x|y^2 + V_{20}x^2 + V_{02}y^2$$
$$+ V_{11}|xy| + \left(\frac{V_{30}V_{20}}{2W_{40}^2} - \frac{V_{30}^3}{8W_{40}^4} - \frac{V_{30}V_{03}^2}{8W_{40}^2W_{04}^2} - \frac{V_{03}^2V_{12}}{16W_{04}^4} + \frac{V_{03}V_{11}}{4W_{04}^2} - \frac{V_{30}V_{03}^2}{16W_{40}W_{04}^3} - 2W_{04} - 2W_{40}\right)|x|$$
$$+ \left(\frac{4V_{03}V_{20} - V_{11}V_{30}}{8W_{40}W_{04}} - \frac{V_{30}^2V_{03}}{8W_{40}^3W_{04}} - \frac{V_{03}^3}{8W_{40}W_{04}^3} + \frac{V_{12}V_{11}}{8W_{04}^2} + \frac{V_{30}^2V_{03}}{32W_{40}^2W_{04}^2} - \frac{V_{03}V_{12}^2}{32W_{04}^4}\right)|y| \quad (61)$$

Condition (58) has to be fulfilled as well.

Substituting (39)–(41), (47)–(49), (56), and (57) into (14) we obtain the resulting formula for the ground state wave function

$$\psi_0(x, y) = \exp\left[-\frac{W_{40}}{3}|x|^3 - W_{04}|x|y^2 - \frac{V_{30}}{4W_{40}}x^2 - \left(\frac{V_{12}}{8W_{04}} - \frac{V_{30}}{8W_{40}}\right)y^2 - \frac{V_{03}}{2W_{04}}|xy| + \left(\frac{V_{30}^2}{8W_{40}^2} + \frac{V_{03}^2}{8W_{40}W_{04}^2} - \frac{V_{20}}{2W_{40}}\right)|x|\right.$$
$$\left. + \left(\frac{V_{30}V_{03}}{16W_{40}W_{04}^2} + \frac{V_{03}V_{12}}{16W_{04}^3} - \frac{V_{11}}{4W_{04}}\right)|y|\right] \quad (62)$$

It is seen that in some cases (62) is quadratically integrable in the whole plane (x, y). The main cases arises if $W_{40} > 0$, $W_{04} > 0$, and $(V_{12}/W_{04}) - (V_{30}/W_{40}) > 0$. This situation can be obtained by appropriate choice of the potential coefficients.

Substituting (47), (48), (56), and (57) into (16) we get an equation for the corresponding ground state energy

$$E_0 = \frac{V_{12}}{4W_{04}} + \frac{V_{30}}{4W_{40}} - \left(\frac{V_{30}^2}{8W_{40}^3} + \frac{V_{03}^2}{8W_{40}W_{04}^2} - \frac{V_{20}}{2W_{40}}\right)^2 - \left(\frac{V_{30}V_{03}}{16W_{40}W_{04}^2} + \frac{V_{03}V_{12}}{16W_{04}^3} - \frac{V_{11}}{4W_{04}}\right)^2 \quad (63)$$

2.3. Solutions of type dI

In this case, we suppose $V_{13} = V_{31} = 0$, $d_{21} \neq 0$, $d_{12} \neq 0$, and from (19) and (20) follows

$$d_{12} = -3d_{30} \quad (64)$$

$$d_{21} = -3d_{03} \quad (65)$$

The resulting wave functions are not quadratic integrable in the whole plane (x, y). The proof will be performed for the quadrant $x \geq 0$, $y \geq 0$ and it can be performed for other quadrants analogously.

Let the wave function of the form (14) be quadratically integrable in the quadrant $x \geq 0$, $y \geq 0$. In this case, it is necessary that $d_{30} > 0$ and $d_{03} > 0$. Substituting (64) and (65) into (14) we get

$$\psi_0(x, y) = h(x, y) = \exp\left[-d_{30}\left(x^3 - 3kx^2y - 3xy^2 + ky^3\right) + \ldots\right] \quad (66)$$

where $k = d_{30}/d_{03} > 0$ and the dots denote lower order terms. To obtain quadratically integrable function (66) in the quadrant $x \geq 0$, $y \geq 0$, it has to have zero limit for all directions going to infinity and lie in the quadrant $x \geq 0$, $y \geq 0$. However, beside the line $x = t$, $y = (k + \sqrt{1 + k^2})t$, this function has a limit

$$\lim_{t\to+\infty} \psi_0\left[x = t, y = \left(k + \sqrt{1 + k^2}\right)t\right] = \lim_{t\to+\infty} \exp\left\{2d_{30}\left[k + \left(1 + k^2\right)^{3/2} + k^3\right]t^3 + \ldots\right\} = +\infty \neq 0 \quad (67)$$

Note, that this case has the $\mathcal{PT}$-symmetric [10] solutions given in ref. 11.

2.4. Solutions of type cII

In this case, it is supposed that $V_{31} = 0$, $V_{13} \neq 0$, $d_{21} = 0$, and $d_{12} \neq 0$. Then (19) is fulfilled automatically and (17), (18), and (20) have the form

$$W_{40}^2 = 9d_{30}^2 \quad (68)$$

$$W_{04}^2 = 9d_{03}^2 + d_{12}^2 \quad (69)$$

$$V_{13} = 12d_{03}d_{12} \quad (70)$$

The solutions of these equations are

$$d_{30} = \frac{W_{40}}{3} \quad (71)$$

$$d_{03} = \pm\frac{1}{6}\sqrt{2W_{04}^2 \pm \sqrt{4W_{04}^4 - V_{13}^2}} \equiv \alpha \quad (72)$$

$$d_{12} = \frac{V_{13}}{12\alpha} \quad (73)$$

To get a quadratically integrable function of the form (14), the first sign in (72) has to be chosen as positive. The second sign can be chosen arbitrarily. These two choices lead to two different potentials and the corresponding ground state wave functions and energies. It is obvious, that a sufficient condition for wave function (14) to be positive is $V_{13} > 0$. Moreover, there exist quadratically integrable solutions that obey more general conditions, but we will not perform a general discussion here.

Equations (21)–(25) can be solved and conditions of their solvability can be obtained in a similar way as was done for the cI class. The result is

$$d_{20} = \frac{V_{30}}{4W_{40}} \quad (74)$$

$$d_{11} = \frac{3V_{21}\alpha}{6W_{40}\alpha + V_{13}} \equiv \beta \quad (75)$$

$$d_{02} = \frac{6V_{03}\alpha - V_{13}\beta}{72\alpha^2} \equiv \gamma \quad (76)$$

$$V_{22} = \frac{V_{13}}{36\alpha^2}(6W_{40}\alpha + V_{13}) \quad (77)$$

$$V_{12} = \frac{V_{13}V_{30}}{12W_{40}\alpha} + \frac{2V_{13}\gamma}{3\alpha} + 6\alpha\beta \quad (78)$$

In these equations, it is supposed that d_{30} is known from (71). If we also suppose that d_{20}, d_{02}, and d_{11} are known from (74) and (75), the solution of (26)–(30) and their conditions of the solvability can be written in the form

$$d_{10} = \frac{V_{20} - \beta^2}{2W_{40}} - \frac{V_{30}^2}{8W_{40}^3} \quad (79)$$

$$d_{01} = \frac{V_{13}}{288W_{40}^3\alpha^2}\left(4W_{40}^2\beta^2 + V_{30}^2 - 4W_{40}^2V_{20}\right) + \frac{V_{02} - \beta^2 - 4\gamma^2}{6\alpha} \quad (80)$$

$$V_{11} = \frac{V_{13}^2V_{30}^2}{864W_{40}^3\alpha^3} + \frac{V_{13}}{18\alpha^2}\left(V_{02} - 4\gamma^2 - \beta^2\right) + \frac{V_{13}^2}{216W_{40}\alpha^3}\left(\beta^2 - V_{20}\right) + \frac{V_{30}\beta}{W_{40}} + 4\beta\gamma \quad (81)$$

$$V_{10} = \left(\frac{V_{30}}{2W_{40}^2} - \frac{V_{13}\beta}{36W_{40}\alpha^2}\right)\left(V_{20} - \beta^2\right) + \frac{V_{13}V_{30}^2\beta}{144W_{40}^3\alpha^2} + \frac{2V_{02}\beta - 8\beta\gamma^2 - V_{13} - 2\beta^3}{6\alpha} - \frac{V_{30}^3}{8W_{40}^4} - 2W_{40} \quad (82)$$

$$V_{01} = \left(\frac{\beta}{W_{40}} - \frac{V_{13}\gamma}{18W_{40}\alpha^2}\right)\left(V_{20} - \beta^2\right) + \frac{V_{13}V_{30}^2\gamma}{72W_{40}^3\alpha^2} + \frac{\gamma}{3\alpha}\left(2V_{02} - 2\beta^2 - 8\gamma^2\right) - \frac{V_{30}^2\beta}{4W_{40}^3} - 6\alpha \quad (83)$$

Substituting $V_{31} = 0$ and (77) and (78) to (15), we get the formula for the potential as

$$V(x,y) = W_{40}^2x^4 + W_{04}^2y^4 + V_{13}|x||y|^3 + \frac{V_{13}}{36\alpha^2}(6W_{40}\alpha + V_{13})x^2y^2 + V_{30}|x|^3 + V_{03}|y|^3 + V_{21}x^2|y| + \left(\frac{V_{13}V_{30}}{12W_{40}\alpha} + \frac{2V_{13}\gamma}{3\alpha} + 6\alpha\beta\right)|x|y^2 + V_{20}x^2 + V_{02}y^2 + V_{11}|xy| + V_{10}|x| + V_{01}|y| \quad (84)$$

Here, the coefficients V_{11}, V_{10}, and V_{01} have to be calculated using (81)–(83) and

$$\alpha = \frac{1}{6}\sqrt{2V_{04} \pm \sqrt{4V_{04}^2 - V_{13}^2}} \quad (85)$$

$$\beta = \frac{3V_{21}\alpha}{6W_{40}\alpha + V_{13}} \quad (86)$$

$$\gamma = \frac{6V_{03}\alpha - V_{13}\beta}{72\alpha^2} \quad (87)$$

Substituting (71)–(75), (79), and (80) into (14), the resulting ground state wave function for the solutions of type cII is obtained in the form

$$\psi_0(x,y) = \exp\left\{-\frac{W_{40}}{3}|x|^3 - \alpha|y|^3 - \frac{V_{13}}{12\alpha}|x|y^2 - \frac{V_{30}}{4W_{40}}x^2 - \gamma y^2 - \beta|xy| + \left(\frac{V_{20} - \beta^2}{2W_{40}} - \frac{V_{30}^2}{8W_{40}^3}\right)|x| + \left[\frac{V_{13}}{288W_{40}^3\alpha^2}\left(4W_{40}^2\beta^2 + V_{30}^2 - 4W_{40}^2V_{20}\right) + \frac{V_{02} - \beta^2 - 4\gamma^2}{6\alpha}\right]|y|\right\} \quad (88)$$

Substituting (74)–(76), (79), and (80) into (16), the formula for the corresponding ground state energy is obtained in the form

$$E_0 = \frac{V_{30}}{2W_{40}} + 2\gamma - \left(\frac{V_{20} - \beta^2}{2W_{40}} - \frac{V_{30}^2}{8W_{40}^3}\right)^2 - \left[\frac{V_{13}}{288W_{40}^3\alpha^2}\left(4W_{40}^2\beta^2 + V_{30}^2 - 4W_{40}^2V_{20}\right) + \frac{V_{02} - \beta^2 - 4\gamma^2}{6\alpha}\right]^2 \quad (89)$$

2.5. Solutions of type dII

Here, we suppose $V_{31} = 0$, $V_{13} \neq 0$, $d_{21} \neq 0$, and $d_{12} \neq 0$. In this case, (17)–(21) have the form

$$V_{40} = 9d_{30}^2 + d_{21}^2 \quad (90)$$

$$V_{04} = 9d_{03}^2 + d_{12}^2 \quad (91)$$

$$0 = 12d_{30}d_{21} + 4d_{21}d_{12} \quad (92)$$

$$V_{13} = 12d_{03}d_{12} + 4d_{12}d_{21} \quad (93)$$

$$V_{22} = 6d_{30}d_{12} + 4d_{12}^2 + 4d_{21}^2 + 6d_{21}d_{03} \quad (94)$$

From (90)–(93) we can find

$$d_{03} = \frac{1}{3}\sqrt{V_{04} - 9d_{30}^2} \quad (95)$$

$$d_{21} = \pm\sqrt{V_{40} - 9d_{30}^2} \quad (96)$$

$$d_{12} = -3d_{30} \quad (97)$$

To get a quadratically integrable wave function of the form (14), the positive sign of the coefficient d_{03} has been chosen. Substituting (95)–(97) into (93) we get

$$V_{13} = \mp 12d_{30}\sqrt{V_{40} - 9d_{30}^2} - 12d_{30}\sqrt{V_{04} - 9d_{30}^2} \quad (98)$$

After squaring, modifying, and performing the substitution $\alpha = 9d_{30}^2$, we obtain

$$\alpha^2 - \frac{V_{04} + V_{40}}{2}\alpha \mp \alpha\sqrt{V_{40} - \alpha}\sqrt{V_{04} - \alpha} + \frac{V_{13}^2}{32} = 0 \quad (99)$$

Equation (99) is an equation of type (B1) (analysed in Appendix B). Here, α is the unknown and

$$d = V_{40} \qquad e = V_{04} \qquad f = \frac{V_{13}^2}{32} \quad (100)$$

and

$$a = V_{40} + V_{04} \qquad b = V_{40}V_{04}$$
$$c = 2\sqrt{V_{40}V_{04} - \frac{V_{13}^2}{16}} \quad (101)$$

Variable α has to be real and positive, because d_{30} has to be real to get a quadratically integrable wave function. We suppose that variables V_{40} and V_{04} are positive (i.e., $d > 0$ and $e > 0$). Using results of Appendix B, it is seen that to get α real and positive, the necessary and sufficient condition is $de > 2f$, that is,

$$V_{40}V_{04} > \frac{V_{13}^2}{16} \quad (102)$$

must be fulfilled.

Possible solutions of (99) are

$$\alpha_1 = \frac{V_{13}^2}{16(V_{40} + V_{04}) + 8\sqrt{16V_{40}V_{04} - V_{13}^2}} \quad (103)$$

$$\alpha_2 = \frac{V_{13}^2}{16(V_{40} + V_{04}) - 8\sqrt{16V_{40}V_{04} - V_{13}^2}} \quad (104)$$

Examining the asymptotic behaviour of (14) in a similar way as in Sect. 2.3 and using (90)–(97), the condition for the wave function to be quadratically integrable can be expressed as

$$8V_{40}d_{21} - 3V_{13}d_{30} - 8(V_{40})^{3/2} > 0 \quad (105)$$

It can be found that if (98) has to be equivalent to (99) and (105) has to be fulfilled, then

$$V_{13} < 0 \quad (106)$$

must be fulfilled.

Next, if $V_{13} < 0$ and if the positive sign is chosen for the coefficient d_{21}, then α_1 solves (98) and (99) and numerical tests indicate that (105) is fulfilled and α_1 leads to a quadratically integrable solution. In some cases, α_2 solves (98) and (99), however numerical tests indicate that α_2 does not lead to a quadratically integrable wave function. In summary, we get the resulting formulae for the wave function coefficients as

$$d_{03} = \frac{1}{3}\sqrt{V_{04} - \alpha_1} \quad (107)$$

$$d_{30} = \frac{1}{3}\sqrt{\alpha_1} \quad (108)$$

$$d_{21} = \sqrt{V_{40} - \alpha_1} \quad (109)$$

$$d_{12} = -\sqrt{\alpha_1} \quad (110)$$

where α_1 is defined by (103).

From (94), (95)–(97), and (99), it is possible to get the condition for the potential coefficient V_{22} as

$$V_{22} = \frac{48V_{40}\alpha_1 - 16V_{04}\alpha_1 + V_{13}^2}{16\alpha_1} \quad (111)$$

Solution of (22)–(25) and the condition of its solvability can be writen in terms of the variables d_{30}, d_{03}, and d_{21} as

$$d_{20} = \frac{V_{30}}{12d_{30}} + \frac{d_{21}}{36d_{30}} \frac{(V_{30}+V_{12})d_{03} + 2V_{03}d_{30}}{2d_{30}^2 - d_{21}d_{03} - d_{03}^2} \tag{112}$$

$$d_{02} = \frac{V_{03}}{12d_{03}} - \frac{d_{30}}{12d_{03}} \frac{(V_{30}+V_{12})d_{03} + 2V_{03}d_{30}}{2d_{30}^2 - d_{21}d_{03} - d_{30}^2} \tag{113}$$

$$d_{11} = -\frac{1}{6} \frac{(V_{30}+V_{12})d_{03} + 2V_{03}d_{30}}{2d_{30}^2 - d_{21}d_{03} - d_{03}^2} \tag{114}$$

$$V_{21} = \frac{d_{30}[3V_{30}(d_{03}+d_{21}) + V_{12}(3d_{03}-d_{21})]}{3\left(2d_{30}^2 - d_{21}d_{03} - d_{03}^2\right)} - \frac{2d_{21}d_{03}(2V_{30}d_{21} + 3V_{30}d_{03} - V_{12}d_{21})}{9d_{30}\left(2d_{30}^2 - d_{21}d_{03} - d_{03}^2\right)} + \frac{V_{03}\left(18d_{30}^2 + d_{21}^2 - 3d_{21}d_{03}\right)}{9\left(2d_{30}^2 - d_{21}d_{03} - d_{03}^2\right)} \tag{115}$$

For the sake of generality, it would be necessary to discuss the value of the term $2d_{30}^2 - d_{21}d_{03} - d_{03}^2 = 0$ in the numerator of these expressions. This simple discussion will not be given here.

Solution of (26)–(30) and conditions of its solvability are written in terms of the wave function coefficients known from the previous equations as

$$d_{10} = \frac{d_{03}\left(3V_{20} - 12d_{20}^2 - 3d_{11}^2\right) - d_{21}\left(V_{02} - 4d_{02}^2 - d_{11}^2\right)}{6d_{30}(d_{21}+3d_{03})} \tag{116}$$

$$d_{01} = \frac{V_{20} + V_{02} - 4d_{20}^2 - 4d_{02}^2 - 2d_{11}^2}{2(d_{21}+3d_{03})} \tag{117}$$

$$V_{11} = 4d_{11}(d_{20}+d_{02}) + \frac{2d_{21}^2}{3d_{30}(3d_{03}+d_{21})}\left(-V_{02} + 4d_{02}^2 + d_{11}^2\right) + \frac{2d_{21}d_{03}}{d_{30}(3d_{03}+d_{21})}\left(V_{20} - 4d_{20}^2 - d_{11}^2\right) + \frac{6d_{30}}{3d_{03}+d_{21}}\left(-V_{02} - V_{20} + 4d_{02}^2 + 4d_{20}^2 + 2d_{11}^2\right) \tag{118}$$

$$V_{10} = \frac{d_{11}}{3d_{03}+d_{21}}\left(V_{20} + V_{02} - 4d_{20}^2 - 4d_{02}^2 - 2d_{11}^2\right) + \frac{2d_{20}d_{03}}{d_{30}(3d_{03}+d_{21})}\left(V_{20} - 4d_{20}^2 - d_{11}^2\right) - \frac{2d_{20}d_{21}}{3d_{30}(3d_{03}+d_{21})}\left(V_{02} - 4d_{02}^2 - d_{11}^2\right) \tag{119}$$

$$V_{01} = -2(3d_{03}+d_{21}) + \frac{2d_{02}}{3d_{03}+d_{21}}\left(V_{20} + V_{02} - 4d_{20}^2 - 4d_{02}^2 - 2d_{11}^2\right) + \frac{d_{03}d_{11}}{d_{30}(3d_{03}+d_{21})}\left(V_{20} - 4d_{20}^2 - d_{11}^2\right) - \frac{d_{21}d_{11}}{3d_{30}(3d_{03}+d_{21})}\left(V_{02} - 4d_{02}^2 - d_{11}^2\right) \tag{120}$$

Note that these fractions are regular, because the case $d_{21} = –3d_{03}$ leads to $V_{13} = 0$ which is not dII class but dI class.

Potentials belonging to class dII and the appropriate ground state wave functions can be obtained as follows. Suppose $V_{31} = 0$. Coefficients V_{40}, V_{04}, and V_{13}, must be chosen to fulfill conditions (102) and (106). Then, α_1 must be determined from (99) and coefficients d_{03}, d_{21}, and d_{12} from (107)–(110). Potential coefficient V_{22} is given by (111). Further, the remaining potential coefficients are given by (115), (118)–(120), and the wave function coefficients follow from (112)–(114), (116), and (117). The resulting wave function of the form (14) is quadratically integrable in the whole plane (x, y). The corresponding ground state energy can be calculated using (16).

The preceding discussion shows that the results in the class dII can also be obtained. However, explicit formulae for the potential, ground state wave function and the corresponding energy are too complex, so that we do not include them here.

3. Conclusion

The idea pursued in this paper is to search for analytic solutions of the two-dimensional Schrödinger equation in cases when other known methods like the separation of variables are unusable. This problem appears to be rather difficult and, for this reason, we have aimed for its partial solution, namely the problem of ground states of two-dimensional fourth-order (quartic) polynomial potential. For the sake of generality, we have used the algebraic method of the solution of the Schrödinger equation. The advantage of this algebraic approach is its generality not relying on any special properties (like symmetry, supersymmetry, etc.) of the problem. Our method is based on generalization of the one-dimensional approach used in refs. 6–8 published in ref. 9. All possible solutions have been found and they have been classified into eight classes, denoted aI, bI, cI, dI, aII, bII, cII, and dII. The most important classes are aI, bI, cI, cII, and dII, because it has been shown that these classes contain physically interesting (quadratically integrable) wave functions. These functions, corresponding energies, and the conditions for the potential have been found.

In Appendix A, it is shown that for any two-dimensional fourth-order polynomial potential it is in general possible to perform the rotation of the coordinates leading to $V_{31} = 0$. Again, this result has been used in the preceding sections.

In Appendix B, (B1), which is needed in the main text, has been analyzed and its solutions together with the conditions of its solvability found.

Generalization of the method to other types of potentials will be a subject of further research.

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Appendix A

A.1. Theorem

Let $V(x,y) = \sum_{0 \le i+j \le 4} V_{ij} x^i y^j$ be a polynomial of a degree less than or equal to four.

Then there exists a rotation in the plane (x, y)

$$\boldsymbol{R} : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{A1}$$

where p and q are real numbers satisfying $p^2 + q^2 = 1$ and there exists a polynomial

$$\tilde{V} = \tilde{V}(x,y) = \sum_{i+j \le 4} \tilde{V}_{ij} x^i y^j \tag{A2}$$

such that

$$\tilde{V}_{31} = 0 \tag{A3}$$

and

$$\forall x, y \in \boldsymbol{R} : \tilde{V}(x,y) = V(\tilde{x}, \tilde{y}) \tag{A4}$$

where $\boldsymbol{R}$ denotes the set of all real numbers. Before the proof, we prove the following lemma.

A.2. Lemma

Let the continuous function $P(t){:}\boldsymbol{R} \to \boldsymbol{R}$ be decomposed to an even continuous function $P^{\text{even}}(t)$ and to an odd continuous function $P^{\text{odd}}(t)$ as

$$P(t) = P^{\text{even}}(t) + P^{\text{odd}}(t) \tag{A5}$$

and let the function $P^{\text{even}}(t)$ have at least one real root. Then $P(t)$ has at least one real root.

A.3. Proof of the lemma

Let t_0 satisfy $P^{\text{even}}(t_0) = 0$. Then also $P^{\text{even}}(-t_0) = 0$, $P(t_0) = P^{\text{odd}}(t_0)$, and $P(-t_0) = -P^{\text{odd}}(t_0)$. Thus, $P(t)$ has a root in the interval $[-t_0, t_0]$.

A.4. Proof of the theorem

From the condition (A4) we obtain

$$\tilde{V}_{31} = \tilde{V}_{31}(p,q) = V_{31}p^4 + (4V_{40} - 2V_{22})p^3 q + (3V_{13} - 3V_{31})p^2 q^2 + (2V_{22} - 4V_{04})pq^3 - V_{13}q^4 \tag{A6}$$

We need to prove that the polynomial $\tilde{V}_{31}(p,q)$ has a real root (p, q) satisfying the condition $p^2 + q^2 = 1$. $\tilde{V}_{31}(p,q)$ is a homogenous polynomial, that is why it is sufficient to prove that $\tilde{V}_{31}(p,q)$ has a real root (p, q) satisfying the condition $p^2 + q^2 > 0$.

We will show that there always exists a real p such that $\tilde{V}_{31}(p,1) = 0$, that is,

$$\tilde{V}_{31}(p,1) = V_{31}p^4 + (4V_{40} - 2V_{22})p^3 + (3V_{13} - 3V_{31})p^2 + (2V_{22} - 4V_{04})p - V_{13} = 0 \tag{A7}$$

Without losing generality we suppose that $V_{31} > 0$.

If

$$P(z) = V_{31}z^2 + (3V_{13} - 3V_{31})z - V_{13} \tag{A8}$$

has a real non-negative root z_1 then the function

$$\tilde{V}_{31}^{\text{even}}(p,1) = V_{31}p^4 + (3V_{13} - 3V_{31})p^2 - V_{13} \tag{A9}$$

has a root (roots) $\pm\sqrt{z_1}$ and

$$\tilde{V}_{31}(p,1) = V_{31}p^4 + (4V_{40} - 2V_{22})p^3 + (3V_{13} - 3V_{31})p^2 + (2V_{22} - 4V_{04})p - V_{13} \tag{A10}$$

has a real root in the interval $[-\sqrt{z_1}, \sqrt{z_1}]$ according to the lemma. If $V_{13} < 0$ then the root

$$\frac{3V_{31} - 3V_{13} + \sqrt{(3V_{31} - 3V_{13})^2 + 4V_{13}V_{31}}}{2} \tag{A11}$$

of the polynomial $P(z)$ is real and positive. If $V_{13} \ge 0$ then $P(0) \le 0$ and $P(z) > 0$ for large positive z or vice versa. Thus, $P(z)$ has a non-negative real root and exist $p_0 \in \boldsymbol{R}$ such that

$$V_{31}p_0^4 + (4V_{40} - 2V_{22})p_0^3 + (3V_{13} - 3V_{31})p_0^2 + (2V_{22} - 4V_{04})p_0 - V_{13} = 0 \tag{A12}$$

and

$$\mathbf{R} = \begin{pmatrix} \frac{p_0}{\sqrt{1+p_0^2}} & \frac{1}{\sqrt{1+p_0^2}} \\ \frac{-1}{\sqrt{1+p_0^2}} & \frac{p_0}{\sqrt{1+p_0^2}} \end{pmatrix} \tag{A13}$$

is the desired rotation.

Appendix B

In this section, we analyse the equation

$$x^2 - \frac{d+e}{2}x \pm x\sqrt{(d-x)(e-x)} + f = 0 \tag{B1}$$

Let us refer to it as a "plus version" if the sign before the third term is + and "minus version" if the sign is –. We suppose that d, e, and f are real numbers and our aim is to find real solutions of (B1).

Moving the term with the square root of (B1) to the right-hand side and squaring both sides of this equation we get the same equation for the plus version as for the minus one

$$\frac{(e-d)^2 + 8f}{4}x^2 - (e+d)fx + f^2 = 0 \tag{B2}$$

B.1. Case $(e - d)^2 + 8f \neq 0$

First, we will suppose that $(e - d)^2 + 8f \neq 0$. Now, quadratic (B2) has two roots

$$x_1 = 2f\frac{d+e-2\sqrt{de-2f}}{(e-d)^2+8f} = \frac{2f}{d+e+2\sqrt{de-2f}} \tag{B3}$$

$$x_2 = 2f\frac{d+e+2\sqrt{de-2f}}{(e-d)^2+8f} = \frac{2f}{d+e-2\sqrt{de-2f}} \tag{B4}$$

It is seen that if $(e - d)^2 + 8f \neq 0$ then $de > 2f$ is the necessary and sufficient condition to make x_1 and x_2 real.

Now we discuss the question of which solution of (B2), x_1 or x_2, solves the plus, minus or no version of the original (B1). This uncertainty arises here because we squared some equations. It is useful to introduce new variables

$$a \equiv d + e \tag{B5}$$

$$b \equiv de \tag{B6}$$

$$c \equiv 2\sqrt{de-2f} \tag{B7}$$

with the corresponding backward transformation

$$d, e = \frac{a \pm \sqrt{a^2-4b}}{2} \tag{B8}$$

$$f = \frac{4b-c^2}{8} \tag{B9}$$

It is easy to find that for any real d, e, and f the term $a^2 - 4b$ is non-negative.

Now, (B1) has the form

$$x^2 - \frac{a}{2}x \pm x\sqrt{x^2-ax+b} + \frac{b}{2} - \frac{c^2}{8} = 0 \tag{B10}$$

and possible solutions are

$$x_1 = \frac{4b-c^2}{4(a+c)} \tag{B11}$$

$$x_2 = \frac{4b-c^2}{4(a-c)} \tag{B12}$$

After substituting (B11) into (B10) we get

$$\frac{\left(4b-c^2\right)\left(c^2+2ca+4b\right)}{(c+a)^2} \pm \frac{4b-c^2}{c+a}\sqrt{\left(\frac{c^2+2ca+4b}{c+a}\right)^2} = 0 \tag{B13}$$

Similarly, after substituting (B12) into (B10) we obtain

$$\frac{\left(4b-c^2\right)\left(c^2-2ca+4b\right)}{(a-c)^2} \pm \frac{4b-c^2}{a-c}\sqrt{\left(\frac{c^2-2ca+4b}{a-c}\right)^2} = 0 \tag{B14}$$

Now it is important to suppose that we work in the real numbers domain. In this case we can write the absolute values instead of the square roots of the squares and can get the following conditions:

- If $4b = c^2$, then x_1 and x_2 solve the both versions of (B1). It can be seen directly from (B3) and (B4), because $4b = c^2$ if and only if $f = 0$ and consequently $x_1 = x_2 = 0$ is a solution to (B1) evidently.
- If $(c^2 + 2ca + 4b)/(a + c) \leq 0$ then x_1 solves the plus version of (B1).
- If $(c^2 + 2ca + 4b)/(a + c) \geq 0$ then x_1 solves the minus version of (B1).
- If $(c^2 - 2ca + 4b)/(a - c) \leq 0$ then x_2 solves the plus version of (B1).
- If $(c^2 - 2ca + 4b)/(a - c) \geq 0$ then x_2 solves the minus version of (B1).

Note that in the complex numbers domain the problem is slightly different. Both versions of (B1) represent one equation with a different choice of the square root branch. The question of which solution of x_1 and x_2 is the right one must be understood as a question of the correspondence to different square root branches.

For the cases discussed in this paper it is also important that if $d > 0$, $e > 0$, and $0 \leq 2f < de$ then $0 < c \leq 2\sqrt{b} \leq a$ and $0 \leq x_{1,2} \leq \min(d, e)$. The proof of this statement is easy.

B.2. Case $(e - d)^2 + 8f = 0$

Now we discuss the case $(e - d)^2 + 8f = 0$. We need not make general discussion because in the cases discussed in the main text, the variable f is non-negative. For $f \geq 0$ only one case exists; when $(e - d)^2 + 8f = 0$. This is the case $d = e$ and $f = 0$.

In this case, we cannot use (B2) because it has the form $0 = 0$. We must start from (B1) again. If $e = d$ and $f = 0$ then this equation has the form

$$x^2 - \mathrm{d}x \pm x\sqrt{(d-x)^2} = 0 \tag{B15}$$

We see that $x = 0$ is one solution. Now we can divide this equation by x and modify it to the form

$$x - d \pm |x - d| = 0 \tag{B16}$$

Further, we see that also

- any $x \leq d$ solves the plus version,
- any $x \geq d$ solves the minus version.