TWO UNCERTAINTY RELATIONS

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Dedicated to Dr. Zdeněk Havlas on the occasion of his 60th birthday.

Heisenberg and Robertson–Schrödinger uncertainty relations for the coordinate and momentum follow from two stronger uncertainty relations. The first uncertainty relation has classical character and its right-hand side can have an arbitrary value greater than or equal to zero. The second uncertainty relation has quantum character and its right-hand side equals $\hbar^2/4$; its existence is related to the existence of the envelop of the wave function. These two uncertainty relations cannot be obviously improved on. The equality sign in the second relation can be achieved for much larger class of the wave functions than in case of the Heisenberg or Robertson–Schrödinger uncertainty relations.
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The Heisenberg uncertainty relation for the coordinate x and momentum p has the well-known form[1]

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \hbar^2/4\,, \tag{1}$$

where

$$\langle(\Delta x)^2\rangle = \int (x - \langle x\rangle)^2 |\psi|^2 \,\mathrm{d}x, \quad \langle(\Delta p)^2\rangle = \int |(\hat{p} - \langle\hat{p}\rangle)\psi|^2 \,\mathrm{d}x, \tag{2}$$

$\hat{p} = -i\hbar(\partial/\partial x)$, $\langle\,\rangle$ denotes the usual quantum-mechanical mean value, integration is carried out from minus infinity to plus infinity and $\hbar$ is the Planck constant.

The wave function ψ can be always written in the form

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$$\psi = \mathrm{e}^{(is_1 - s_2)/\hbar}, \tag{3}$$

where $s_1 = s_1(x,t)$ and $s_2 = s_2(x,t)$ are real functions of the coordinate x and time t and $\int |\psi|^2 \, \mathrm{d}x = 1$. Using this expression we get[2–5]

$$\langle (\Delta p)^2 \rangle = \langle (\Delta p_1)^2 \rangle + \langle (\Delta p_2)^2 \rangle, \tag{4}$$

where

$$\langle (\Delta p_1)^2 \rangle = \int \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right)^2 |\psi|^2 \, \mathrm{d}x, \quad \langle (\Delta p_2)^2 \rangle = \int \left(\frac{\partial s_2}{\partial x} \right)^2 |\psi|^2 \, \mathrm{d}x \, . \tag{5}$$

We can see that the mean square deviation of the momentum $\langle (\Delta p)^2 \rangle$ can be split into two parts.

The first part $\langle (\Delta p_1)^2 \rangle$ can be interpreted within the statistical generalization of classical mechanics in which the classical momentum $p = \partial S/\partial x_{\mathrm{cl}}$, where S is the classical action and x_{cl} the classical coordinate, is replaced by $\partial s_1/\partial x$ and the probability density $|\psi|^2 = \exp(-2s_2/\hbar)$ is introduced. In agreement with this argument the mean value of the momentum operator $-i\hbar(\partial/\partial x)$ can be written as[5]

$$\langle p \rangle = \int \frac{\partial s_1}{\partial x} |\psi|^2 \, \mathrm{d}x \, . \tag{6}$$

The second part $\langle (\Delta p_2)^2 \rangle$ is proportional to one of the most important quantities appearing in mathematical statistics, the Fisher information I[6–11]

$$I = \int \frac{1}{\rho} \left(\frac{\partial \rho}{\partial x} \right)^2 \mathrm{d}x = \frac{4}{\hbar^2} \int \left(\frac{\partial s_2}{\partial x} \right)^2 |\psi|^2 \, \mathrm{d}x = \frac{4}{\hbar^2} \langle (\Delta p_2)^2 \rangle, \tag{7}$$

where $\rho = |\psi|^2$ is the distribution function. By using the Schwarz inequality $(u,u)(v,v) \geq |(u,v)|^2$, where $(u,v) = \int_{-\infty}^{\infty} u^* v \, \mathrm{d}x$, u and v are complex functions and the star denotes the complex conjugate, it is easy to derive[4,5] the inequality known from mathematical statistics

$$\int (x-a)^2 \rho \, \mathrm{d}x \, I \geq 1 \, , \tag{8}$$

where a is a real number.

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Now we show that the Heisenberg uncertainty relation can be replaced by two uncertainty relations for $\langle(\Delta p_1)^2\rangle$ and $\langle(\Delta p_2)^2\rangle$ (see also[2–5]).

First, we take

$$u = \Delta x|\psi|, \quad v = \left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)|\psi|. \tag{9}$$

Then, the Schwarz inequality yields the first uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p_1)^2\rangle \geq \left[\int \Delta x\left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)|\psi|^2\,\mathrm{d}x\right]^2. \tag{10}$$

It can be shown[4,5] that the function $\partial s_1/\partial x$ in the last integral corresponds to the classical momentum $\partial S/\partial x_{\text{cl}}$ and this relation has the usual meaning known from mathematical statistics: The product of variances of two quantities is greater than or equal to the square of their covariance. Depending on the functions s_1 and s_2, the square of the covariance of the coordinate and momentum at the right-hand side of this relation can have arbitrary values greater than or equal to zero. In this sense, this relation can be denoted as "classical".

The second uncertainty relation can be obtained in an analogous way for

$$u = \Delta x|\psi|, \quad v = \left(\frac{\partial s_2}{\partial x} - \left\langle\frac{\partial s_2}{\partial x}\right\rangle\right)|\psi| \tag{11}$$

with the result

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle \geq \left[\int (x - \langle x\rangle)\left(\frac{\partial s_2}{\partial x} - \left\langle\frac{\partial s_2}{\partial x}\right\rangle\right)|\psi|^2\,\mathrm{d}x\right]^2. \tag{12}$$

The right-hand side of this relation can be simplified[2–5] and yields the second uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle \geq \frac{\hbar^2}{4}. \tag{13}$$

This uncertainty relation follows from the Schwarz inequality in a similar way as the first one, however, the covariance (u,v) is in this case constant and equals $\hbar/2 > 0$ independently of the concrete form of the function s_2. We note also that relation (*13*) is for $\langle x \rangle = a$ equivalent to inequality (*8*) for the Fisher information. Uncertainty relation (*13*) can be understood as the standard statistical inequality, too. However, because of the specific form of the covariance (u,v) which equals $\hbar/2$ independently of s_2, the left-hand side of this relation must be greater than or equal to $\hbar^2/4$. In contrast to the first uncertainty relation, uncertainty relation (*13*) can be denoted as "quantum".

It can be shown[4,5] that the sum of uncertainty relations (*10*) and (*13*) is equivalent to the Robertson–Schrödinger uncertainty relation for the coordinate and momentum[12–14]. The Heisenberg uncertainty relation can be obtained from this sum by neglecting the first term on its right-hand side. Therefore, uncertainty relations (*10*) and (*13*) are stronger than the corresponding Heisenberg and Robertson–Schrödinger uncertainty relations.

The equality sign in uncertainty relations (*10*) and (*13*) is obtained if the real functions s_1 and s_2 are quadratic functions of x of the form $\alpha x^2 + \beta x + \gamma$, where real coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ can depend on time[4,5]. It is worth to notice that this condition for the second uncertainty relation (*13*) is independent of the form of the function s_1 and depends only on s_2, i.e., the envelop of the wave function. Therefore, the equality sign in this relation can be obtained for much larger class of the wave functions than in case of the Heisenberg or the Robertson–Schrödinger uncertainty relations. To find conditions for such states, we assume the wave function ψ in form of Eq. (*3*), where $s_1 = s_1(x,t)$ and $s_2 = s_2(x,t) = \alpha(t)x^2 + \beta(t)x + \gamma(t)$, where $\alpha(t) > 0$. Substituting this wave function into the Schrödinger equation $i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2 + V\psi$, where $V = V(x,t)$ is potential energy, we obtain two equations for the functions s_1 and s_2

$$-\frac{\hbar}{2m}\frac{\partial^2 s_1}{\partial x^2} + \frac{1}{m}\frac{\partial s_1}{\partial x}\frac{\partial s_2}{\partial x} + \frac{\partial s_2}{\partial t} = 0 \tag{14}$$

and

$$\frac{\partial s_1}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial s_1}{\partial x}\right)^2 - \left(\frac{\partial s_2}{\partial x}\right)^2 + \hbar\frac{\partial^2 s_2}{\partial x^2}\right] + V = 0 \tag{15}$$

An example of the wave function obeying these conditions is given in the following section.

FREE PARTICLE

We assume that the wave function of a free particle is at time t = 0 described by the gaussian wave packet combined with the plane wave[4,5]

$$\psi(x,0) = \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-x^2/(2a^2)+ikx} \tag{16}$$

with the energy

$$E = \frac{\hbar^2}{4ma^2} + \frac{\hbar^2 k^2}{2m} , \tag{17}$$

where $a > 0$ and k are real constants. By solving the time Schrödinger equation we get

$$\psi(x,0) = \frac{1}{\sqrt{a\sqrt{\pi}}} \frac{\sqrt{1 - \frac{i\hbar t}{ma^2}}}{\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}} \times$$

$$\times \exp\left\{ - \frac{\left(x - \frac{\hbar k}{m} t\right)^2}{2a^2\left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} + i\left[\frac{kx + \frac{\hbar t x^2}{2ma^4} - \frac{\hbar k^2}{2m} t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2}\right]\right\} . \tag{18}$$

The corresponding functions s_1 and s_2 equal

$$s_1(x,t) = \hbar k \frac{x + \frac{\hbar t x^2}{2ma^4 k} - \frac{\hbar k}{2m} t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2} - \hbar \arctan \frac{\hbar t}{ma^2} \tag{19}$$

and

$$s_2(x,t) = \frac{\hbar}{2}\left\{\frac{\left(x - \frac{\hbar k}{m}t\right)^2}{a^2\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right]} - \ln\frac{1}{a\sqrt{\pi}\sqrt{1+\left(\frac{\hbar t}{ma^2}\right)^2}}\right\}. \tag{20}$$

The mean square deviations of the coordinate and momentum are given by the equations

$$\langle(\Delta x)^2\rangle = \frac{a^2}{2}\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right] \tag{21}$$

and

$$\langle(\Delta p_1)^2\rangle = \frac{\hbar^4 t^2}{2m^2 a^6\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right]}, \quad \langle(\Delta p_2)^2\rangle = \frac{\hbar^2}{2a^2\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right]}. \tag{22}$$

With increasing time, $\langle(\Delta p_1)^2\rangle$ increases to the limit value $\hbar^2/(2a^2)$ and $\langle(\Delta p_2)^2\rangle$ goes down to zero. Therefore, uncertainty relation (*13*) for $\langle(\Delta p_2)^2\rangle$ yields for this example much better estimate than the corresponding Heisenberg uncertainty relation.

The left-hand side and the right-hand side of uncertainty relation (*10*) have the same value

$$\langle(\Delta x)^2\rangle\langle(\Delta p_1)^2\rangle = \left\langle \Delta x\left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)\right\rangle^2 = \frac{\hbar^4 t^2}{4m^2 a^4}. \tag{23}$$

Therefore, the first uncertainty relation is fulfilled with the equality sign at all times.

Calculating the left-hand side of the second uncertainty relation we obtain

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle = \frac{\hbar^2}{4} \tag{24}$$

and see that uncertainty relation (*13*) is fulfilled with the equality sign at all times, too.

For this example, the equality sign in the Heisenberg uncertainty relation is obtained for $t = 0$ only. In case of the Robertson–Schrödinger uncertainty relation, the equality sign is obeyed at all times, however, the right-hand side of the relation increases with time. These results confirm that uncertainty relations (*10*) and (*13*) are stronger than the corresponding Heisenberg and Robertson–Schrödinger uncertainty relations.

CONCLUSIONS

Heisenberg and Robertson–Schrödinger uncertainty relations for the coordinate and momentum known from quantum mechanics follow from two stronger uncertainty relations (*10*) and (*13*).

Uncertainty relation (*10*) can be understood as the inequality for the product of variances of the deviation of the coordinate x and momentum represented by the function $p = \partial s_1/\partial x$ from their mean values which must be greater than or equal to the square of the covariance of these quantities.

Uncertainty relation (*13*) is equivalent to the inequality for the Fisher information and has similar origin as the more general Rao–Cramér inequalities known from mathematical statistics[7]. This relation can be also understood as the inequality between the variances and covariance of the deviation of the coordinate x and the function $\partial s_2/\partial x$ from their mean values. However, the corresponding covariance is constant and equals $\hbar/2$. The square of the covariance then yields the constant $\hbar^2/4$ appearing in the uncertainty relations.

The constant $\hbar^2/4$ in all the uncertainty relations discussed above is related to the existence of the envelop of the wave function. The function s_1, i.e., the real part of the phase of the wave function, is not in this respect important. Therefore, uncertainty relation (*13*) has, in contrast to (*10*), "quantum" character. Due to two separate relations having "classical" and "quantum" character and use of the Schwarz inequality, these two uncertainty relations cannot be obviously improved on.

The equality sign in uncertainty relation (*13*) can be achieved for much larger class of the wave functions than in case of the Heisenberg uncertainty relation (squeezed states) or the Robertson–Schrödinger uncertainty relation. It is important from the experimental point of view as well as from the point of view of the most efficient information transfer.

Finally we note that results of this paper are in full agreement with general discussion of the Fisher information and uncertainty relations discussed in[15–18].

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