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Internal Structure of the Heisenberg and Robertson-Schrödinger Uncertainty Relations

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Abstract It is known that the Heisenberg and Robertson-Schrödinger uncertainty relations can be replaced by sharper uncertainty relations in which the "classical" (depending on the gradient of the phase of the wave function) and "quantum" (depending on the gradient of the envelope of the wave function) parts of the variances $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ are separated. In this paper, three types of uncertainty relations for a different number of classical parts (2, 1 or 0) with different time behaviour of their left-hand and right-hand sides are discussed. For the Gaussian wave packet and two classical parts, the left-hand side of the corresponding relations increases for $t \to \infty$ as t^2 and is much larger than $\hbar^2/4$. For one classical part, the left-hand side of the corresponding relation goes to the right-hand side equal to $\hbar^2/4$. For no classical part, both the right-hand and left-hand sides of the corresponding relation go quickly to zero. Therefore, the well-known limitations following from the usual uncertainty relations can be overcome in the corresponding measurements.

Keywords Quantum mechanics · Uncertainty relations · Three types of uncertainty relations

1 Introduction

The Heisenberg uncertainty relation for the coordinate x and momentum p has the well-known form [1]

$$\big\langle(\Delta x)^2\big\rangle\big\langle(\Delta p)^2\big\rangle \geq \frac{\hbar^2}{4}, \tag{1}$$

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where

$$\left\langle(\Delta x)^2\right\rangle = \int_{-\infty}^{\infty}\big(x-\langle x\rangle\big)^2|\psi|^2\mathrm{d}x, \tag{2}$$

$$\left\langle(\Delta p)^2\right\rangle = \int_{-\infty}^{\infty}\big|\big(\hat{p}-\langle\hat{p}\rangle\big)\psi\big|^2\mathrm{d}x, \tag{3}$$

$\psi = \psi(x,t)$ is the normalized wave function, $\hat{p} = -i\hbar(\partial/\partial x)$, $\langle\,\rangle$ denotes the usual quantum-mechanical mean value and $\hbar$ is the reduced Planck constant $\hbar = h/(2\pi)$. For recent discussion of uncertainty relations see e.g. [2–15]. Detailed discussion of the uncertainty relations in stochastic mechanics can be found in [16].

The normalized wave function ψ can be always written in terms of its modulus and argument (phase)

$$\psi = |\psi|\mathrm{e}^{i\arg(\psi)} = \mathrm{e}^{-s_2/\hbar}\mathrm{e}^{is_1/\hbar}, \tag{4}$$

where $s_1(x,t)$ and $s_2(x,t)$ are real functions. Then we get

$$\hat{p}\psi = \frac{\partial s_1}{\partial x}\psi + i\frac{\partial s_2}{\partial x}\psi. \tag{5}$$

The mean momentum can be written as

$$\langle\hat{p}\rangle = \langle\psi|\hat{p}\psi\rangle = \int_{-\infty}^{\infty}\frac{\partial s_1}{\partial x}|\psi|^2\,\mathrm{d}x + i\int_{-\infty}^{\infty}\frac{\partial s_2}{\partial x}|\psi|^2\,\mathrm{d}x. \tag{6}$$

Assuming the bound states with the property

$$x^n|\psi|^2 \to 0, \quad n = 0, 1, 2,\ x \to \pm\infty \tag{7}$$

the second integral in Eq. (6) does not contribute to the mean momentum

$$\int_{-\infty}^{\infty}\frac{\partial s_2}{\partial x}|\psi|^2\mathrm{d}x = -\frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\mathrm{e}^{-2s_2/\hbar}\mathrm{d}x = -\frac{\hbar}{2}\mathrm{e}^{-2s_2/\hbar}\Big|_{x=-\infty}^{\infty} = 0. \tag{8}$$

Therefore, the resulting expression for the mean momentum [17–20]

$$\langle\hat{p}\rangle = \int_{-\infty}^{\infty}\frac{\partial s_1}{\partial x}|\psi|^2\,\mathrm{d}x \tag{9}$$

does not depend on $\partial s_2/\partial x$. This formula corresponds to transition from the point particle in classical mechanics where the probability density has the δ-like character to the particle described by the probability density $|\psi|^2$ in quantum mechanics. At the same time, the expression for the classical momentum $p_{cl} = \partial S/\partial x$, where S is the Hamilton action is replaced here by the mean value $\langle\hat{p}\rangle = \langle\partial s_1/\partial x\rangle$, where the function s_1 corresponds to S and the probability density $|\psi|^2$ is introduced.

As an example, we consider the wave function of a free particle [17–19] in form of the Gaussian wave packet at time $t = 0$

$$\psi(x,0) = \frac{1}{\sqrt{a\sqrt{\pi}}}\mathrm{e}^{-x^2/(2a^2)+ikx} \tag{10}$$

with the mean energy

$$E = \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2}{4ma^2}, \tag{11}$$

where $a > 0$, k is a real constant and m is the mass of the particle. Solving the time Schrödinger equation we get [17, 18, 20]

$$\psi(x,t) = \frac{1}{\sqrt{a\sqrt{\pi}}} \frac{\sqrt{1-i\alpha}}{\sqrt{1+\alpha^2}} \exp\left[-\frac{(x-\frac{\hbar k}{m}t)^2}{2a^2(1+\alpha^2)} + i\left(\frac{kx + \frac{\alpha x^2}{2a^2} - \frac{\hbar k^2}{2m}t}{1+\alpha^2}\right)\right], \tag{12}$$

where $\alpha = \hbar t/(ma^2)$. The corresponding functions s_1 and s_2 and their derivatives equal

$$s_1(x,t) = \hbar k \frac{x + \frac{\alpha x^2}{2a^2 k} - \frac{\hbar k}{2m}t}{1+\alpha^2} - \hbar \arctan \alpha, \tag{13}$$

$$s_2(x,t) = \frac{\hbar}{2}\left[\frac{(x-\frac{\hbar k}{m}t)^2}{a^2(1+\alpha^2)} - \ln \frac{1}{a\sqrt{\pi}\sqrt{1+\alpha^2}}\right] \tag{14}$$

and

$$\frac{\partial s_1}{\partial x} = \hbar k \frac{1 + \frac{\alpha x}{a^2 k}}{1+\alpha^2}, \tag{15}$$

$$\frac{\partial s_2}{\partial x} = \frac{\hbar(x - \frac{\hbar k}{m}t)}{a^2(1+\alpha^2)}. \tag{16}$$

As it could be anticipated, the mean momentum in this case equals

$$\langle \hat{p} \rangle = \left\langle \frac{\partial s_1}{\partial x} \right\rangle = \hbar k. \tag{17}$$

Relation between the mean coordinate and momentum

$$\langle \hat{x} \rangle = \frac{\hbar k}{m} t = \frac{\langle \hat{p} \rangle}{m} t \tag{18}$$

agrees with the Ehrenfest theorems and is the same as in classical mechanics. In agreement with Eq. (9), the constant a determining the width of the wave packet and $\partial s_2/\partial x$ does not appear in the last two equations.

2 Square of the Momentum and Two Uncertainty Relations

Now, we will discuss $\langle \hat{p}^2 \rangle$ [17–20]. It follows from Eq. (5) that the mean value $\langle \hat{p}^2 \rangle$ can be written as a sum of two parts

$$\langle \hat{p}^2 \rangle = \langle \hat{p}\psi | \hat{p}\psi \rangle = \langle \hat{p}_1^2 \rangle + \langle \hat{p}_2^2 \rangle, \tag{19}$$

where

$$\langle \hat{p}_1^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\partial s_1}{\partial x}\right)^2 |\psi|^2 \, dx \tag{20}$$

and

$$\langle \hat{p}_2^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\partial s_2}{\partial x} \right)^2 |\psi|^2 \, dx. \tag{21}$$

Similarly to our discussion of $\langle \hat{p} \rangle$ in Eq. (9), the first part $\langle \hat{p}_1^2 \rangle$ can be interpreted as statistical generalization of the expression $p_{cl}^2 = (\partial S/\partial x)^2$ from classical mechanics in which the classical momentum $p_{cl} = \partial S/\partial x$ is replaced by $\partial s_1/\partial x$ and the probability density $|\psi|^2$ is introduced. The second part $\langle \hat{p}_2^2 \rangle$ is given by $|\psi|^2$ or the envelope of the wave function $|\psi| = \exp(-s_2/\hbar)$ and does not depend on $\partial s_1/\partial x$. For the Gaussian wave packet, Eq. (19) leads to energy (11).

It is obvious that such separation applies not only for $\langle \hat{p}^2 \rangle$ and kinetic energy but also for the variance $\langle (\Delta \hat{p})^2 \rangle$ appearing in the Heisenberg uncertainty relation [17–23]

$$\langle (\Delta \hat{p})^2 \rangle = \langle \left(\hat{p} - \langle \hat{p} \rangle \right)^2 \rangle = \langle (\Delta \hat{p}_1)^2 \rangle + \langle (\Delta \hat{p}_2)^2 \rangle, \tag{22}$$

where

$$\langle (\Delta \hat{p}_1)^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\partial s_1}{\partial x} - \langle \hat{p} \rangle \right)^2 |\psi|^2 \, dx \tag{23}$$

and

$$\langle (\Delta \hat{p}_2)^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle \right)^2 |\psi|^2 \, dx = \int_{-\infty}^{\infty} \left(\frac{\partial s_2}{\partial x} \right)^2 |\psi|^2 \, dx. \tag{24}$$

We note that interesting approach to the classical-quantum decomposition of variances can be found in [24].

It was shown that Heisenberg uncertainty relation (1) can be replaced by two sharper relations for $\langle (\Delta \hat{p}_1)^2 \rangle$ and $\langle (\Delta \hat{p}_2)^2 \rangle$ [17–23]. These relations can be obtained from the Schwarz inequality

$$\langle u|u \rangle \langle v|v \rangle \geq \left| \langle u|v \rangle \right|^2, \tag{25}$$

where u and v are complex functions, $\langle u|v \rangle = \int_{-\infty}^{\infty} u^* v \, dx$ and the star denotes the complex conjugate.

Taking the functions

$$u = \Delta x |\psi| = \left(x - \langle x \rangle \right) |\psi| \tag{26}$$

and

$$v = \left(\frac{\partial s_1}{\partial x} - \langle \hat{p} \rangle \right) |\psi| \tag{27}$$

we get the first relation

$$\langle (\Delta x)^2 \rangle \langle (\Delta \hat{p}_1)^2 \rangle \geq \left[\int_{-\infty}^{\infty} \Delta x \left(\frac{\partial s_1}{\partial x} - \langle \hat{p} \rangle \right) |\psi|^2 dx \right]^2. \tag{28}$$

This relation has the usual meaning known from mathematical statistics: The product of variances of two quantities is greater than or equal to the square of their covariance. Depending on the functions $\partial s_1/\partial x$ and $|\psi|^2$, the square of the covariance of the coordinate and momentum at the right-hand side of this relation can have arbitrary values greater than or equal to zero. In this sense, this inequality has classical character.

The second relation can be obtained in an analogous way for

$$u = \Delta x|\psi| \tag{29}$$

and

$$v = \frac{\partial s_2}{\partial x}|\psi| \tag{30}$$

with the result

$$\big\langle(\Delta x)^2\big\rangle\big\langle(\Delta\hat{p}_2)^2\big\rangle \geq \left[\int_{-\infty}^{\infty}\big(x - \langle x\rangle\big)\frac{\partial s_2}{\partial x}|\psi|^2\mathrm{d}x\right]^2. \tag{31}$$

The last integral can be calculated as

$$\begin{aligned} &\int_{-\infty}^{\infty}\big(x - \langle x\rangle\big)\frac{\partial s_2}{\partial x}|\psi|^2\mathrm{d}x \\ &\quad = -\frac{\hbar}{2}\int_{-\infty}^{\infty}\big(x - \langle x\rangle\big)\frac{\partial|\psi|^2}{\partial x}\mathrm{d}x \\ &\quad = -\frac{\hbar}{2}\left[\big(x - \langle x\rangle\big)|\psi|^2\big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}|\psi|^2\mathrm{d}x\right] = \frac{\hbar}{2}, \end{aligned} \tag{32}$$

where the expression $|\psi|^2 = \exp(-2s_2/\hbar)$, integration by parts, boundary conditions (7) and the normalization condition for the wave function are used. The resulting uncertainty relation

$$\big\langle(\Delta x)^2\big\rangle\big\langle(\Delta\hat{p}_2)^2\big\rangle \geq \frac{\hbar^2}{4} \tag{33}$$

follows from the Schwarz inequality in a similar way as relation (28) and can be understood as the standard statistical inequality, too. However, the covariance $\langle u|v\rangle$ equals in this case $\hbar/2$, does not depend on the concrete form of the functions s_1 and s_2 and the left-hand side of this relation cannot equal zero. In contrast to relation (28), the left-hand side of relation (33) does not depend on s_1 and depends only on s_2 giving the envelope $|\psi| = \exp(-s_2/\hbar)$ of the wave function ψ. In this sense, relation (33) has quantum character.

The sum of relations (28) and (33)

$$\big\langle(\Delta x)^2\big\rangle\big\langle(\Delta\hat{p})^2\big\rangle \geq \left[\int_{-\infty}^{\infty}\Delta x\left(\frac{\partial s_1}{\partial x} - \langle\hat{p}\rangle\right)|\psi|^2\mathrm{d}x\right]^2 + \frac{\hbar^2}{4} \tag{34}$$

is equivalent to the Robertson-Schrödinger uncertainty relation for the coordinate and momentum [18, 25–27]. The Heisenberg uncertainty relation can be obtained from this sum by neglecting the first expression appearing at the right-hand side of relation (34). Therefore, relations (28) and (33) are sharper than the corresponding Heisenberg and Robertson-Schrödinger uncertainty relations.

3 Momentum Representation

It is evident that analogous approach can be used also for the wave function in the momentum representation

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)\,\mathrm{e}^{px/(i\hbar)}\mathrm{d}x \tag{35}$$

that can be written in form analogous to Eq. (4)

$$\varphi(p) = \mathrm{e}^{-r_2/\hbar}\mathrm{e}^{ir_1/\hbar}, \tag{36}$$

where $r_1(p)$ and $r_2(p)$ are real functions. Using the coordinate operator in the momentum representation $\hat{x} = i\hbar(\partial/\partial p)$ it is easy to derive equations analogous to Eqs. (9) and (19)–(24)

$$\langle\hat{x}\rangle = -\int_{-\infty}^{\infty}\frac{\partial r_1}{\partial p}|\varphi|^2\,\mathrm{d}p, \tag{37}$$

$$\langle\hat{x}^2\rangle = \langle\hat{x}\varphi|\hat{x}\varphi\rangle = \langle\hat{x}_1^2\rangle + \langle\hat{x}_2^2\rangle, \tag{38}$$

$$\langle(\Delta\hat{x})^2\rangle = \langle(\hat{x} - \langle\hat{x}\rangle)^2\rangle = \langle(\Delta\hat{x}_1)^2\rangle + \langle(\Delta\hat{x}_2)^2\rangle, \tag{39}$$

where

$$\langle\hat{x}_1^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial r_1}{\partial p}\right)^2|\varphi|^2\,\mathrm{d}p, \tag{40}$$

$$\langle\hat{x}_2^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial r_2}{\partial p}\right)^2|\varphi|^2\,\mathrm{d}p, \tag{41}$$

$$\langle(\Delta\hat{x}_1)^2\rangle = \int_{-\infty}^{\infty}\left(-\frac{\partial r_1}{\partial p} - \langle\hat{x}\rangle\right)^2|\varphi|^2\,\mathrm{d}p \tag{42}$$

and

$$\langle(\Delta\hat{x}_2)^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial r_2}{\partial p} - \left\langle\frac{\partial r_2}{\partial p}\right\rangle\right)^2|\varphi|^2\,\mathrm{d}p = \int_{-\infty}^{\infty}\left(\frac{\partial r_2}{\partial p}\right)^2|\varphi|^2\,\mathrm{d}p. \tag{43}$$

Relations in the momentum representation corresponding to Eqs. (28) and (33) have the form

$$\langle(\Delta\hat{x}_1)^2\rangle\langle(\Delta p)^2\rangle \geq \left[\int_{-\infty}^{\infty}\left(-\frac{\partial r_1}{\partial p} - \langle\hat{x}\rangle\right)\Delta p|\varphi|^2\mathrm{d}p\right]^2 \tag{44}$$

and

$$\langle(\Delta\hat{x}_2)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4}. \tag{45}$$

Comments to these relations can be made similar as in case of relations (28) and (33) and will not be given here. In the above mentioned sense, the first relation (44) has classical character and the second one (45) has quantum character.

4 Relation for Two Quantum Parts

It has been shown above that both the mean values $\langle(\Delta\hat{x})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ can be expressed as a sum of two parts having classical and quantum character. Now we ask, if it is possible to derive some relation for the quantum parts of $\langle(\Delta\hat{x})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ only, without the presence of parts having classical character. Such relation is discussed in this section.

First, we use the same expression in the coordinate representation as in Eq. (30)

$$u = \frac{\partial s_2}{\partial x} |\psi| = -\hbar \frac{\partial |\psi|}{\partial x} \tag{46}$$

and the Fourier transform of an analogous expression $-\hbar(\partial|\varphi|/\partial p)$ in the momentum representation

$$v = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(-\hbar \frac{\partial|\varphi|}{\partial p}\right) \mathrm{e}^{-px/(i\hbar)} \mathrm{d}p, \tag{47}$$

where φ is the Fourier transform of the wave function ψ given by Eq. (35). We get

$$\langle u|u\rangle = \big\langle (\Delta \hat{p}_2)^2 \big\rangle, \tag{48}$$

$$\langle v|v\rangle = \big\langle (\Delta \hat{x}_2)^2 \big\rangle \tag{49}$$

and

$$\langle u|v\rangle = \hbar^2 \int_{-\infty}^{\infty} \frac{\partial|\psi|}{\partial x} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{\partial|\varphi|}{\partial p} \mathrm{e}^{-px/(i\hbar)} \mathrm{d}p\, \mathrm{d}x. \tag{50}$$

Then, Schwarz inequality (25) yields the relation for the quantum parts of $\langle(\Delta\hat{x})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$

$$\big\langle (\Delta \hat{x}_2)^2 \big\rangle \big\langle (\Delta \hat{p}_2)^2 \big\rangle \geq \big|\langle u|v\rangle\big|^2 = \hbar^2 |I|^2, \tag{51}$$

where

$$I = \hbar \int_{-\infty}^{\infty} \frac{\partial|\psi|}{\partial x} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{\partial|\varphi|}{\partial p} \mathrm{e}^{-px/(i\hbar)}\, \mathrm{d}p\, \mathrm{d}x. \tag{52}$$

In contrast to uncertainty relations (1), (33) and (45), the right-hand side of this relation is not in general equal to $\hbar^2/4$ and the value of the integral I has to be calculated in each case separately. If the wave functions ψ and φ depend also on time, all quantities in relation (51) may be time dependent. For this reason, we do not denote Eq. (51) as an uncertainty relation but as a relation or an inequality.

The wave functions ψ and φ are related by the Fourier transform. In contrast, this is not generally the case for their envelopes $|\psi|$ and $|\varphi|$. For this reason, relation (51) has different character than the usual uncertainty relations and can lead to interesting results.

To give an example, we assume that $|\varphi|$ equals the Fourier transform of $|\psi|$. Using Eq. (35) we then get

$$I = -i \int_{-\infty}^{\infty} \frac{\partial|\psi(x)|}{\partial x} \left\{ \int_{-\infty}^{\infty} \big|\psi(x')\big| \left[\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \mathrm{e}^{p(x'-x)/(i\hbar)} \mathrm{d}p \right] x' \mathrm{d}x' \right\} \mathrm{d}x. \tag{53}$$

Taking into account that the expression in brackets equals the Dirac δ-function we have

$$I = -i \int_{-\infty}^{\infty} \frac{\partial|\psi|}{\partial x} x |\psi|\, \mathrm{d}x. \tag{54}$$

Performing integration by parts we get

$$\int_{-\infty}^{\infty} \frac{\partial|\psi|}{\partial x} x|\psi| \mathrm{d}x = x\,|\psi|^2\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} |\psi| \left(x \frac{\partial|\psi|}{\partial x} + |\psi| \right) \mathrm{d}x. \tag{55}$$

It follows from boundary conditions (7) and the normalization condition for the wave function ψ that

$$\int_{-\infty}^{\infty} |\psi(x)| x \frac{\partial |\psi(x)|}{\partial x} \mathrm{d}x = -\frac{1}{2}. \tag{56}$$

Therefore, if $|\varphi|$ equals the Fourier transform of $|\psi|$ relation (51) has the usual right-hand side

$$\langle (\Delta \hat{x}_2)^2 \rangle \langle (\Delta \hat{p}_2)^2 \rangle \geq \frac{\hbar^2}{4}. \tag{57}$$

For example, it is valid for the ground state of the linear harmonic oscillator.

If $|\varphi|$ does not equal the Fourier transform of $|\psi|$, integral I has to be calculated in each case separately. As shown in the following section, it leads to interesting results.

5 Gaussian Wave Packet

In this section, we return back to our example discussed in Introduction.

The quantities appearing in relations (1), (28), (33), (34), (44), (45) and (51) equal in this case

$$\langle (\Delta \hat{x})^2 \rangle = \langle (\Delta \hat{x}_1)^2 \rangle + \langle (\Delta \hat{x}_2)^2 \rangle, \tag{58}$$

$$\langle (\Delta \hat{x}_1)^2 \rangle = \frac{a^2 \alpha^2}{2}, \tag{59}$$

$$\langle (\Delta \hat{x}_2)^2 \rangle = \frac{a^2}{2}, \tag{60}$$

$$\langle (\Delta \hat{p})^2 \rangle = \langle (\Delta \hat{p}_1)^2 \rangle + \langle (\Delta \hat{p}_2)^2 \rangle, \tag{61}$$

$$\langle (\Delta \hat{p}_1)^2 \rangle = \frac{\hbar^2 \alpha^2}{2a^2(1+\alpha^2)} \tag{62}$$

and

$$\langle (\Delta \hat{p}_2)^2 \rangle = \frac{\hbar^2}{2a^2(1+\alpha^2)}, \tag{63}$$

where $\alpha = \hbar t/(ma^2)$.

Using these results, we discuss now the left-hand and right-hand sides of relations (1), (28), (33), (34), (44), (45) and (51).

First we see that $\langle (\Delta \hat{x}_1)^2 \rangle \langle (\Delta \hat{p}_1)^2 \rangle$ and also $\langle (\Delta \hat{x})^2 \rangle \langle (\Delta \hat{p})^2 \rangle$ in relations (1) and (34) increase for $t \to \infty$ as $\hbar^2 \alpha^2/4 \approx t^2$. Therefore, the left-hand side of these relations increases in time due to the presence of two classical parts $\langle (\Delta \hat{x}_1)^2 \rangle$ and $\langle (\Delta \hat{p}_1)^2 \rangle$ at their left-hand side.

Products $\langle (\Delta \hat{x}_2)^2 \rangle \langle (\Delta \hat{p}_1)^2 \rangle$, $\langle (\Delta \hat{x}_1)^2 \rangle \langle (\Delta \hat{p}_2)^2 \rangle$ and also $\langle (\Delta \hat{x}_2)^2 \rangle \langle (\Delta \hat{p})^2 \rangle$, $\langle (\Delta \hat{x})^2 \rangle \times \langle (\Delta \hat{p}_2)^2 \rangle$ in relations (33) and (45) go for $t \to \infty$ to $\hbar^2/4$. Therefore, if there is only one classical part at the left-hand side as in case of relations (33) and (45), the left-hand side tends to $\hbar^2/4$.

In contrast to these cases, the left-hand side of relation (51) given by Eqs. (60) and (63)

$$\langle (\Delta \hat{x}_2)^2 \rangle \langle (\Delta \hat{p}_2)^2 \rangle = \frac{\hbar^2}{4(1+\alpha^2)} \tag{64}$$

does not contain the classical parts and behaves for $t \to \infty$ as $1/t^2$. This behaviour of relation (51) is very different from relations (1), (28), (33), (34), (44) and (45). It implies that also the right-hand side of relation (51) must go to zero for $t \to \infty$.

Calculating the inner product $\langle u|v\rangle$ appearing at the right-hand side of Eq. (51) we get

$$\langle u|v\rangle = -i\frac{\sqrt{2}\hbar a^2}{(2+\alpha^2)^{5/2}}\left(1+\alpha^2\right)^{1/4}\left[k^2\left(1+2\alpha^2+i\alpha^3\right)-\left(2+\alpha^2\right)/a^2\right] \times \mathrm{e}^{-k^2a^2(2\alpha^2-2i\alpha+1)/(2+\alpha^2)/2}. \tag{65}$$

The corresponding formula for the right-hand side $|\langle u|v\rangle|^2$ of relation (51) can be calculated easily and will not be given here.

Special interesting cases are

$$\left|\langle u|v\rangle\right|^2 = \frac{\hbar^2}{4}\frac{\sqrt{1+\alpha^2}}{(1+\alpha^2/2)^3}, \quad k=0 \tag{66}$$

and

$$\left|\langle u|v\rangle\right|^2 = \frac{\hbar^2}{4}\left(k^2a^2/2-1\right)^2\mathrm{e}^{-k^2a^2/2}, \quad t=0. \tag{67}$$

For $k=0$ and $t=0$, the last two equations lead to the well-known value

$$\left|\langle u|v\rangle\right|^2 = \frac{\hbar^2}{4} \tag{68}$$

appearing in Eqs. (1), (33), (34) and (45).

The right-hand side $|\langle u|v\rangle|^2$ of Eq. (51) given by Eq. (66) decreases to zero for $t \to \infty$. The characteristic value of α for which $|\langle u|v\rangle|^2$ in Eq. (66) becomes significantly smaller than $\hbar^2/4$ is $\alpha \approx 1$. For the electron mass and a typical nanoscale dimension $a = 10^{-9}$ m, the condition $\alpha \approx 1$ is fulfilled already for very short time $t_0 \approx 10^{-14}$ s. For $t \gg t_0$, the right-hand side of relation (51) goes to zero and this relation represents practically no restriction on the possible values of the product $\langle(\Delta\hat{x}_2)^2\rangle\langle(\Delta\hat{p}_2)^2\rangle$ at the left-hand side.

Similarly, the characteristic value of ka for which $|\langle u|v\rangle|^2$ in Eq. (67) becomes smaller than $\hbar^2/4$ is given by the condition $ka \approx 1$. For $a = 10^{-9}$ m, this condition is fulfilled for $k_0 \approx 10^9$ m^{-1}. Therefore, the right-hand side of relation (51) is for the wave vectors $k \gg k_0$ close to zero.

We note also that for $k \neq 0$, $a \neq 0$ and $t \to \infty$ the right-hand side of Eq. (51) given by Eq. (65)

$$\left|\langle u|v\rangle\right|^2 \approx \frac{2a^{10}k^4m^3\mathrm{e}^{-2a^2k^2}}{\hbar t^3} \tag{69}$$

goes to zero as $1/t^3$, i.e. faster than its left-hand side (64).

The constant $\hbar^2/4$ appears at the right-hand side of relations (1), (33), (34) and (45) due to the presence of classical terms $\langle(\Delta\hat{x}_1)^2\rangle$ and/or $\langle(\Delta\hat{p}_1)^2\rangle$ at the left-hand side of these relations. In contrast to these relations, the right-hand side of relation (51) goes to zero.

The width of the Gaussian wave packet in the coordinate or momentum representation measured by $\langle(\Delta\hat{x})^2\rangle$ or $\langle(\Delta\hat{p})^2\rangle$ increases in time and the wave packet spreads out. In contrast to it, quantities $\langle(\Delta\hat{x}_2)^2\rangle$ and $\langle(\Delta\hat{p}_2)^2\rangle$ appearing in relation (51) do not increase in time and their product goes to zero.

Quantities $\langle(\Delta\hat{x}_2)^2\rangle$ and $\langle(\Delta\hat{p}_2)^2\rangle$ are given by the mean values of the square of the derivatives $\partial s_2/\partial x$ and $\partial r_2/\partial p$, where s_2 and r_2 give the probability densities $|\psi|^2 = \exp(-2s_2/\hbar)$ and $|\varphi|^2 = \exp(-2r_2/\hbar)$ in the coordinate and momentum representation, respectively. Measurement of the probability densities in the coordinate and momentum representation should be feasible for example for the photon wave packets. Therefore, relation (51) is interesting also from the experimental point of view.

6 Conclusion

In this paper, internal structure of the Heisenberg and Robertson-Schrödinger uncertainty relations has been investigated.

It is known that the mean square deviation of the momentum in the coordinate representation from its mean value can be written as a sum of the classical and quantum parts, Eq. (22). Similar result is valid also for the coordinate in the momentum representation, Eq. (39). The classical parts in these expressions depend on the phase of the wave function and probability density in the corresponding representation. Function s_1 giving the phase of the wave function in the coordinate representation corresponds to the Hamilton action. The quantum parts depend on the probability densities or envelopes of the wave functions only and have not their counterpart in classical mechanics.

A few relations for the classical and quantum parts have been discussed. Depending on the number of classical parts in the relations, three types of the relations have been introduced:

(I) Heisenberg and Robertson-Schrödinger uncertainty relations (1) and (34) have two classical parts at their left-hand side and $\hbar^2/4$ (and additional square of the covariance term in case of the Robertson-Schrödinger relation) at their right-hand side. For the Gaussian wave packet and $t \to \infty$, their left-hand sides behave as t^2.

The left-hand sides of relations (28) and (44) contain two classical parts, too. For the Gaussian wave packet and $t \to \infty$, the left-hand sides of these relations have the same behaviour as in case of the Heisenberg and Robertson-Schrödinger uncertainty relations. The right-hand sides contain the square of the covariance term appearing also in the Robertson-Schrödinger uncertainty relation (34) and can equal zero.

(II) Relations (33) and (45) have only one classical part at their left-hand side and $\hbar^2/4$ at the right-hand side. For the Gaussian wave packet and $t \to \infty$, the left-hand sides tend to $\hbar^2/4$.

(III) The left-hand side of relation (51) contains the quantum parts only and behaves for the Gaussian wave packet and $t \to \infty$ as $1/t^2$. Except for special cases, the right-hand side is not equal to $\hbar^2/4$ and has to be calculated in each case separately. For the Gaussian wave packet and $t \to \infty$, it behaves as $1/t^3$ (see Eq. (69)).

The sum of relations (28) and (33) yields the Robertson-Schrödinger uncertainty relation (34). Therefore, relations (28) and (33) are sharper than the Heisenberg and Robertson-Schrödinger uncertainty relations. In contrast to relation (33), the right-hand side of relation (28) does not contain $\hbar^2/4$ and can equal zero. Similar conclusions can be made also for analogous relations (44) and (45) in the momentum representation.

It is seen from our example that the time dependence of relations I–III can be very different:

With increasing time, the left-hand sides of relations I containing two classical parts behave for the Gaussian wave packet and $t \to \infty$ as t^2 and the right-hand sides are greater than or equal to $\hbar^2/4$. From the experimental point of view, it limits attainable accuracy of measurements.

Uncertainty relations II contain one classical part only and their left-hand sides go for the Gaussian wave packet and $t \to \infty$ to the right-hand side $\hbar^2/4$. From the experimental point of view, this situation is more favourable than in case I.

In case III, the left-hand side of relation (51) for the Gaussian wave packet and $t \to \infty$ goes to zero as $1/t^2$ and the right-hand side as $1/t^3$. Therefore, accuracy of measurement of the product of two quantum parts $\langle(\Delta\hat{x}_2)^2 = \langle(\partial r_2/\partial p)^2\rangle$ and $\langle(\Delta\hat{p}_2)^2 = \langle(\partial s_2/\partial x)^2\rangle$ is not in general case limited by $\hbar^2/4$ as in cases I and II. Relation (51) is only the usual relation between the variances and covariance of two quantities valid for any measurement.

Reason that Heisenberg and Robertson-Schrödinger uncertainty relations (1) and (34) and relations (33) and (45) contain $\hbar^2/4$ is the existence of the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ valid in the coordinate as well as in the momentum representation (see also calculation in Eqs. (53)–(56)). In contrast to this situation, the inner product $\langle u|v\rangle$ in relation (51) depends on the derivatives $\partial|\psi|/\partial x$ and $\partial|\varphi|/\partial p$ of the functions $|\psi|$ and $|\varphi|$ that are not, except for special cases, related by the Fourier transform. In this respect, relation (51) is different from the Heisenberg and Robertson-Schrödinger uncertainty relations and relations (28), (33), (44) and (45).

Our example shows that relations II and III are less restrictive than uncertainty relations I. Therefore, the well-known limitations following from the usual uncertainty relations can be overcome in measurements related to the envelopes of the wave function in the coordinate and momentum representations. Such measurements should be feasible for example for the photon wave packets.

These results clarify the internal structure of the Heisenberg and Robertson-Schrödinger uncertainty relations and show that these relations can be replaced by sharper relations with different time behaviour of their left-hand and right-hand sides. It is not only of theoretical interest but also important from the experimental point of view.

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