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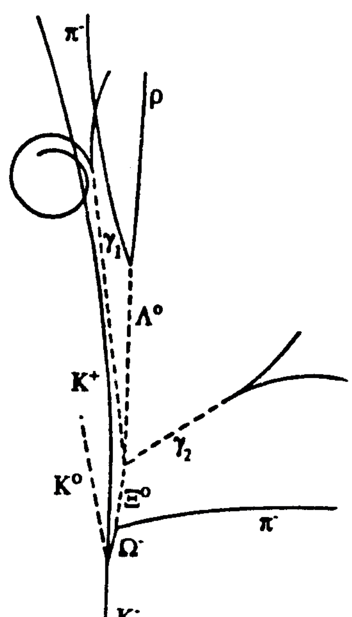

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THE NON-EXISTENCE OF OBSERVABLE STATES WITH FRACTIONAL CHARGE IN CP ^{2} MODELS

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The $\mathbb{CP}^2$ model is examined with regard to the theoretical basis for the non-existence of observable states with fractional charge in the strong interactions. It is demonstrated that the field variables in quantum chromodynamics can be formulated on this this space, which arises both through gravity and the unified gauge theory and the existence of composite states which transform trivially under $SU(3)$ is verified.

Key words and phrases. complex projective space, instanton-anti-instanton pair, composite states, linear potential.

1. Introduction

It has been noted that the $\mathbb{CP}^2$ sigma model appears to approximate well the nonperturbative aspects of quantum chromodynamics. This will be investigated further here in the context of the coset spaces that arise in the reduction of higher-dimensional field theories to the coupling of standard model and gravity. It is found that the consistency of the hypothesis of the existence only of states transforming under the trivial representation of SU(3) is confirmed through the properties of the projective space.

In the second section, two contributions to the potential of the strong interaction are are discussed. The quark-anti-quark pair can be regarded as a composite state with a restricted set of interactions. Together with the Hamiltonion of the electric dipole, the perturbative sum of the ladder diagrams may be evaluated. Furthermore, the instanton-anti-instanton effects may viewed as a nonperturbative solution to the equations of a σ-model in complex projective space. Again, a linear potential is derived, and the two contributions may be added to give the quark-anti-quark potential. The role of complex projective space in the higher-dimensional geometry of unified theories which produce the standard model upon reduction to four dimensions is summarized in the third section.

2. The Complex Projective Space and Quantum Chromodynamics

The complex projective space $\mathbb{CP}^2$ as the target space of a σ-model [1] or a topological fluctuation of space-time [2]. The Euclidean gravitational field equations with a cosmological term Λ are known to have $\mathbb{CP}^2$ are known to have the $\mathbb{CP}^2$ instanton as a solution. It has volume $\frac{18\pi^2}{\Lambda^2}$ and action $\frac{9\pi}{4\Lambda}$ [2]. Since it has positive curvature, it would be weighted as less probable than flat space. Nevertheless, there can exist tunneling to this vacuum solution.

It has been noted that a Coulomb term in the potential can be generated with the strength of the interaction determined by the perturbative series in quantum chromodynamics. Convergence of the perturbations series requires a lower bound for the isounit of the hadrons [3]. Relating it to the isospin, and combining this with an integrality of the sum of charges of the quarks, it can be demonstrated that the quark charges equal the known values [4].

The necessity for a $q\bar{q}$ meson or qqq baryon, rather than a free quark, can be deduced if the lower bound for the isounit, upon identification with the isospin, is larger than $\frac{1}{2}$. The divergence of the perturbation series for lower values of $|I|$ describing free quarks would reflect the lack of observability of fractional charges.

In addition to this term, the potential of quantum chromodynamics is known to be the absolute value of a linear function of the distance. The physical effect of this potential is the restriction of quark fields to a region of nuclear dimensions. The potential has a basis in the string models of hadrons. If two quarks are attached to the ends of the string, this would be classical force field would be a monotonically increasing function of the distance. The tunneling probability for free quarks has been proven to vanish [5].

Nonperturbative effects in quantum chromodynamics have been modelled by a $\mathbb{CP}^2$ field theory. By contrast with the symmetry group $SU(3)$, $\pi_2(SU(3)/U(2)) = \mathbb{Z}$ allowing the existence of instantons. The $\mathbb{C}^{N-1}$ model with fermions is

$$I = \int d^2x \Big((D_\mu z)^\dagger D_\mu z + \psi^\dagger (\not{D} - M)\psi \frac{1}{2N}(g_s + e^2 f)(\psi^\dagger \gamma_\mu \psi)^2 - \frac{g_v}{2N}[(\psi^\dagger \tau^i \psi)^2 + (\psi^\dagger \gamma_5 \tau^i \psi)^2] \Big) \tag{2.1}$$

where $D_\mu z_\alpha = \partial_\mu z_\alpha - \frac{f}{N}\left(z^\dagger \overleftrightarrow{\partial}_\mu z\right) z_\alpha$ and $D_\mu \psi^a_\alpha = \partial_\mu \psi^a_\alpha - e\frac{f}{N}(z^\dagger \overleftrightarrow{\partial}_\mu z)\psi^a_\alpha$, τ^i generate the flavour symmetry of the ψ fields and $z^\dagger z = \frac{N}{2f}$, $z^\dagger \psi = \psi^\dagger z = 0$ [6] [7]. In a $\frac{1}{N}$ expansion, the quartic terms in z are eliminated through the introduction of two new auxiliary fields $\alpha(x)$ and $\lambda^\mu(x)$ such that the propagator is demonstrated to have a pole at zero momentum indicating a long range force that confines the z and ψ fields.

The supersymmetric model is

$$I = \int d^2x \Big[(D_\mu z_\alpha)^\dagger D_\mu z_\alpha - i(\psi^\dagger_\alpha \not{D} \psi_\alpha) + \frac{1}{4}[(\psi^\dagger_\alpha \psi_\alpha)^2 + (\psi^\dagger_\alpha \gamma_5 \psi_\alpha)^2 - (\psi^\dagger_\alpha \gamma_\mu \psi_\alpha)^2] \Big] \tag{2.2}$$

with $z^\dagger_\alpha z_\alpha = 1$ and $z^\dagger_\alpha \psi_\alpha = \psi^\dagger_\alpha z_\alpha = 0$ which is invariant under local $U(1)$ transformations and supersymmetry transformations. It is known that the pole in the λ^μ propagator is not present in the supersymmetric model and that confinement of the z and ψ fields does not occur [7].

The connection between the Wilson loop variables, $\mathbb{CP}^2$ and quantum chromodynamics has been explored previously. The area law has been derived for a Wilson loop variable

(2.3)

$$\langle W^C[A]\rangle_{YM} = \left\langle \left\langle exp\left[ig \oint_C dx^u n^A V_\mu^A(x)\right]\right\rangle_{pert.\ YM} \cdot exp\left[i \oint_C \omega\right]\right\rangle_{TQFT}$$

where $n^A = \langle \Lambda|\xi^\dagger T^A \xi|\Lambda\rangle$, $|\Lambda\rangle$ is a highest-weight coherent state of SU(3) corresponding to the coset space $SU(3)/U(2)$, T^A is a generator of SU(3), $\omega = \langle \Lambda|i\xi^\dagger d\xi(x)|\Lambda\rangle$, the perturbative factor is dominated by an Abelian projected effective gauge theory, where the Abelian group is a subgroup of $U(2)$ and the non-perturbative part is defined by a four-dimensional topological quantum field theory [8]. Because this topological quantum field theory admits a BRST symmetry, it can be reduced through a superspace integration to a two-dimensional bosonic σ model. It is found that the contribution of instanton-anti-instanton interactions to the Wilson loop variable yields an exponential dependence with respect to the negative of the area contained by the curve C [9]. This result is consistent with confinement of nontrivial gluon states.

Supersymmetry has been shown to be necessary for finiteness of the model, and yet, it is not compatible with confinement of quarks and gluons. It is also known that Borel nonsummability of certain one-dimensional quantum mechanical models is related to the interactions of instantons and anti-instantons [10][11]. It follows that quark and gluon confinement will not be a property of the theory of the strong interactions based on the $\mathbb{CP}^2$ model unless there are divergences in the perturbation series that are not being removed through supersymmetry. These divergences require an evaluation of the partition function for quantum fluctuations about a nonperturbative state, which would depend on the relative energy of the instanton-anti-instanton pair to the vacuum.

It is known that the perturbative series of quantum chromodynamics is not Borel summable because of infrared renormalons and instanton singularities [12]. While the renormalons tend to obscure the connection with nonperturbative effects [13], these can be removed by passing to the conformal limit [14]. However, these instanton singularities are generated by solutions of Yang-Mills field equations that do not yield an immediate prediction of the confinement of quarks. For example, the potential derived from the interaction of QCD instantons and anti-instantons has the form

$$V(R) = V_{Coul}(R) - \int d^3x_I \frac{d\rho}{\rho^5} n(\rho)[v_0(\rho) + v_2(\rho)] \tag{2.4}$$

where $\vec{x}_I$ denotes the location of the instanton, ρ is the instanton scale size, $n(\rho)$ is the instanton density, $v_0(R)$ is a classical term and $v_2(\rho)$ is found from diagrams with one-gluon exchange and self-energy graphs [15][16]. The linear term in the potential is not recovered, however, unless the point-particle picture is replaced by interactions of nonperturbative states defined by the $\mathbb{CP}^2$ model. Specifically, mapping of the two-dimensional areas of the σ-model to the complex projective space yield interactions in higher dimensions. If the charge of the instanton and anti-instanton is distributed throughout the surface of $\mathbb{CP}^2$, for example, an integration would yield a factor of R^6, which would cancel the factor arising from the integration of $\frac{1}{\rho^5}$, because $v_0(\rho)$ and $v_2(\rho)$ produce a factor of $\frac{1}{R}$, to give a linear potential when the quark-anti-quark separation is related to the size of the $\mathbb{CP}^2$ spaces.

The existence of a $q\bar{q}$ pair separated by a distance r is nonperturbative especially if the Hamiltonian is determined by a Wilson loop. Any perturbative computation based on this configuration is a correction to a non-perturbative state. The potential for quantum chromodynamics also has been computed through a sum of ladder diagrams [17], which follows from the interactions that can occur between $q\bar{q}$ states. The combinatoric factor $\frac{2n!}{n!(n+1)!}$ is reduced because there are restrictions on the planar pair-wise contractions of $2n$ vertices.

The potential deduced through the $\mathbb{CP}^2$ model may be added to that computed perturbatively about the nonperturbative $q\bar{q}$ state to give the potential for the strong interactions. At a distance R from a colour electric dipole, the expectation value of the square of the electric field is $\langle E^2 \rangle \propto \frac{d^2}{R^6} + \ldots$, where d is the distance between the quark and the anti-quark [18]. Although it might appear energitically favourable for the two charges to move in the orthogonal direction to the separation vector between the quark and anti-quark, if this configuration develops, an instanton-anti-instanton interaction results, and integration over the surface areas yields a compensating factor of R^6. Consequently, a linear potential at larger distances is valid generally.

It might be concluded that since the Lagrangian of quantum chromodynamics with fermions is included in the perturbative factor and the ghost action that is equivalent to the topological field theory, there is no need to extend the bosonic sigma model that results from the Parisi-Sourlas dimensional reduction Quark confinement also follows from the linear potential, which may be derived through the logarithm of the Wilson loop variable, and it is attained with the bosonic σ model. A two-dimensional σ-model with fermions would have to be derived from a four-dimensional action to have a connection with chromodynamics. Although terms corresponding to the fermion fields in the σ-model might be added to the topological field theory, the Parisi-Sourlas reduction cannot be implemented if the model is not supersymmetric. Generalizing from two dimensions to four dimensions,

$$I_4 = \int d^4x \Big[D_\mu \bar{z} D_\mu z + \bar{\psi}(\gamma^\mu D_\mu - M_B)\psi + \frac{f}{2N} e^2 (\bar{\psi}\gamma_\mu \psi)^2 + \frac{g_V}{24}[(\bar{\psi}\bar{z}\overleftrightarrow{\partial}^\mu z\psi)^2 + +(\bar{\psi}\bar{z}\overleftrightarrow{\partial}^\mu \gamma_5 z\psi)^2] \Big] \tag{2.5}$$

where the scalar fields are identified with gluons, $|z|^2 = \frac{n}{2f}$ and $D_\mu = \partial_\mu - \frac{2ef}{N}\bar{z}\overleftrightarrow{\partial}^\mu z$ [19]. While the internal space is $\mathbb{CP}^2$, the integral is evaluated over M^4. There does not exist a spin structure on $\mathbb{CP}^2$, and the fermion fields could not be extended to a total space which is locally diffeomorphic to $M^4 \times \mathbb{CP}^2$. Instead, the extension of fermion fields to the total space of the bundle over Minkowski space-time will require a generalized spin-structure which depends on phase factors containing the gauge field [20]. Integration of the partition function yields an effective action that contains determinant of the trace of the logarithms of the differential operators and an integral containing auxiliary fields that are necessary for the elimination of the quartic field terms. The auxiliary fields have the form of trivial representations of $SU(3)$. The quark and gluon fields are not present and there remain only the hadron fields in the effective action [19]. This mechanism therefore provides a theoretical basis for the non-existence of free fractional charge states in the strong interactions. The action (3.4) is not renormalizable [19], although there is a supersymmetric version of the four-dimensional σ-model [21].

The $SU(3)$ symmetry of the $\mathbb{CP}^2$ model is global in contrast to the local invariance of quantum chromodynamics. These first invariance can be regarded as an approximation to the latter symmetry in the limit of infinitesimal radius of an internal space. The extension to higher dimensions would be necessary for the initial formulation of this model to be valid. There, a generalized spin structure involving the product of the fermion field and a gauge field phase factor is required. This phase factor can be identified with an open string. Furthermore, it is known that consistency requires that the fermions transform under the representation of a double covering of the group acting on the bosonic fields [20]. Although $SO(6)$ is larger than the isometry group $SU(3)$, it does act on S^6 in the other coset compactification. Furthermore, it has a double covering $SU(4)$, which might be considered to be a preceding invariance of the effective field theory before the $SU(3)$ symmetry of quantum chromodynamics. The projection of $SU(4)$ to $SU(3)$ and $SO(6)$ to $SU(3)/\mathbb{Z}_2$ verifies the necessary constraint on these fields such that a generalized spin structure may be defined.

It has been found that the instantons of the $\mathbb{CP}^1$ and $\mathbb{CP}^2$ models are located below a melting point for the exponential suppression of the mass of the η' meson [22], whereas there are no instanton solutions to the $\mathbb{CP}^n$ are available for modelling phenomena in quantum chromodynamcs because these are above this critical point for large field fluctuations. This result would confirm the use of the $\mathbb{CP}^2$ model.

3. The Complex Projective Space in a Compactification of Extra Dimensions

It may be recalled that there exists a coset space $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$ [23], which is eight-dimensional and can be retracted through the groups to $\frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1)' \times U(1)''}$ [24][25] such that the known particle spectrum is recovered and the automorphism group of the spinor space of the standard model is obtained [26]. The reduction sequence from a twelve-dimensional theory compactified over this space to a four-dimensional theory with the gauge group $SU(3) \times SU(2) \times U(1)$ has been shown to be consistent with known higher-dimensional models that are verified by particle physics phenomenology at lower energies.

The seven-dimensional manifolds $M_{klm} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1)' \times U(1)''}$, which are solutions to the equations of eleven-dimensional supergravity, have an eight-dimensional limit $\frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1)'}$, with the $U(1)'$ and $U(1)''$ factors identified, which admits

$N = 2$ supersymmetry and contains the lepton and quark multiplets of the standard model [25]. The space $\frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$ is classified by the Chern class of the $U(1)$ bundle over $\mathbb{CP}^2$ and $\mathbb{CP}^1$, where $\mathbb{CP}^2 \simeq SU(3)/(SU(2) \times U(1))$.

Consequently, the complex projective spaces arise in the reduction sequence from higher dimensions and may be used as a basis for theoretical models of elementary particles. The isometry group of $\mathbb{CP}^2$ is $SU(3)/\mathbb{Z}_3$ [2] [27], and the discrete group commutes with $SU(3)$ and the embedding of $SU(2) \times U(1)$. Both the covering of the isometry group of $\mathbb{CP}^2$, $SU(3)$, and the stability group, $SU(2) \times U(1)$, occur in the standard model in four dimensions.

In the reduction sequence corresponding to the coset space $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$, the nonperturbative bound states of the vector bosons of the strong interactions have a special role and take values in S^7 which then can be projected to $SU(2) \times SU(2)$. There is a mapping between nonperturbative configurations in S^7 to $SU(3)/U(1)$. These states are represented by Wilson loop variables in S^7, which may be identified as closed strings, whereas open strings on $SU(3)/U(1)$ is required for the modelling of the hadrons. With a further $SU(2)$ invariance, projection to $SU(3)/(SU(2)\times U(1)) \simeq \mathbb{CP}^2$ would yield the open string states on $\mathbb{CP}^2$ which could serve as the phase factors necessary to introduce the fermion fields. The existence of the open string states requires a discrete factorization which results from the presence of $\mathbb{Z}_3$ in the isometry group. Factorization by the restriction to an order-two subgroup produces the strings with a quark and anti-quark, whereas the entire $\mathbb{Z}_3$ group would generate the open string description of baryons.

The compactification of the twelve-dimensional theory, containing the ten-dimensional superstring theories, based on the coset space $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$, has been shown to be constrained by the presence of $N = 1$ supersymmetry and the number of fermion generations. It has been shown that the radius of the six-dimensional component $G_2/SU(3)$ [28] then equals approximately

$$R_0 \approx 0.7224\ \ell_P. \tag{3.1}$$

Since the contraction of the isometry and stability groups in this coset manifold yield the bundle over $\mathbb{CP}^2$, it follows that the radius of $\mathbb{CP}^2$ would be similarly constrained. Furthermore, the $\mathbb{CP}^2$ gravitational instanton arising as a topological fluctuation of the metric is likely to have dimensions of the same magnitude of a Planck length. With such an infinitesimal radius of this compact space, the approximation of a local SU(3) symmetry is sufficient.

4. Conclusion

There are several nonperturbative aspects of a $\mathbb{CP}^2$ model that reflect composite states in quantum chromodynamics. This model consistently predicts the the monotonically increasing potential and the absence of fractionally charged states outside of this potential. The existence of a generalized spin structure, necessary for the introduction of fermion fields, in the higher-dimensional space is valid if there is a coupling to a phase factor representing an open string state.

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ON THE PROLONGATIONS OF REPRESENTATIONS OF LIE ALGEBRAS

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Abstract

In our previous paper, we have studied prolongations of representations of Lie groups. In this paper, we present a study on the prolongations of representations of Lie algebras. We show that a tangent bundle of a given Lie algebra attains a Lie algebra structure. Then, we prove that this tangent bundle is algebraically isomorphic to the Lie algebra of a tangent bundle of a Lie group. Using these, we define prolongations of representations of Lie algebras.

Keywords: Prolongation; Representation; Lie Algebra.

Introduction

In this paper, we present a study on the prolongations of representations of Lie algebras. In our previous work [6], we have obtained a basis for the tangent bundle of an arbitrary finite-dimensional vector space and shown that if a function is linear, then its tangent function is also linear. We have also defined prolongations of finite-dimensional real representations of Lie groups and obtained faithful representations on tangent bundles of Lie groups. The remaining part of the paper was dedicated to the study of the properties of these faithful representations.

We start this study based on the fact that existence of prolongations of representations of Lie algebras can be implied by the existence of prolongations of representations of Lie groups [1, 9]. First we show that the tangent bundle of a Lie algebra has a Lie algebra structure. Then, we prove that this tangent bundle is algebraically isomorphic to $Lie(TG)$ where G represents a Lie group, TG represents G's tangent bundle, and $Lie(TG)$ represents the Lie algebra of TG. Using these, we define prolongations of representations of Lie algebras.

1 Preliminaries

THEOREM 1.1. *For manifolds* M *and* N, $T(M)\times T(N)$ *is equivalent to* $T(M\times N)$ *by using the following relation*

$$(X,Y)\cong Tf_x(Y)+T\bar{f}_y(X) \tag{1.1}$$

for all $X\in T_x(M)$ *and* $Y\in T_y(M)$, *where* $f_x : N\to M\times N$ *and* $\bar{f}_y : M\to M\times N$ *defined by* $f_x(m)=(x,m)$ *and* $\bar{f}_y(m)=(m,y)$, *where* $T_x(M)$ *represents the tangent space of* M *at* $x\in M$.

DEFINITION 1.2. If we consider a coordinate neighborhood U in M with a local coordinate system $\{x_1,x_2,...,x_n\}$, then we can canonically define a local coordinate system

$\{x_1,x_2,...,x_n,v_1,v_2,...,v_n\}$ on $T(U)$, i.e., a tangent vector $\sum_{i=1}^{n} v_i(\frac{\partial}{\partial x_i})_x$ has the coordinates $(x_1, x_2, ..., x_n, v_1, v_2, ..., v_n)$ if the point $x \in U$ has the coordinates $(x_1, x_2, ..., x_n)$. This local coordinate system $\{x_1, x_2, ..., x_n, v_1, v_2, ..., v_n\}$ is called the induced local coordinate system on $T(U)$ by $\{x_1, x_2, ..., x_n\}$ [3, 8].

Definition 1.3. If we consider two tangent vectors $X \in T_x(\mathbb{R}^n)$ and $Y \in T_y(\mathbb{R}^n)$, then the tangent bundle $T(\mathbb{R}^n)$ is a vector space of dimension $2n$ with respect to the following sum "$\oplus$" and the scalar multiplication "$\bullet$"

$$\begin{aligned} X \oplus Y &= (T\tau_y)X + (T\tau_x)Y, \\ \lambda \bullet X &= (T\sigma_\lambda)X \end{aligned} \tag{1.2}$$

where τ_x represents a translation of $\mathbb{R}^n$ by $x \in \mathbb{R}^n$ and σ_λ represents a the scalar multiplication by $\lambda \in \mathbb{R}$. For any finite-dimensional vector space V, the tangent bundle $T(V)$ becomes a vector space with respect to the similar sum and the scalar multiplication. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $Tf : T(\mathbb{R}^n) \to T(\mathbb{R}^m)$ is also a linear map [7].

Proposition 1.4. *Let V and W be arbitrary finite-dimensional real vector spaces of dimensions n, m and $f : V \to W$ be a linear map, then the tangential map $Tf : TV \to TW$ is a linear function. Moreover if f is a linear isomorphism, then Tf is also a linear isomorphism.[6]*

Definition 1.5. Let V be an n-dimensional real vector space, $\{\alpha_i : 1 \leq i \leq n\}$ be a basis for V, $\{e_i : 1 \leq i \leq n\}$ be the standard basis for $\mathbb{R}^n$ and ψ be bundle trivialization of TV. Then ψ is a linear isomorphism with respect to the both vector space structure on $V \times \mathbb{R}^n$ and the structure on TV and if we define $\bar{\alpha}_i = (\alpha_i, 0) \in V \times \mathbb{R}^n$ and $y_i = (0, e_i) \in V \times \mathbb{R}^n$ for $\forall i \in \{1, 2, ..., n\}$, then by definition $\eta = \{\bar{\alpha}_i, y_i : 1 \leq i \leq n\}$ is a basis for $V \times \mathbb{R}^n$. Since ψ is a linear isomorphism, then

$$\psi^{-1}(\eta) = \{\widetilde{\alpha}_i, \widetilde{y}_i : \widetilde{\alpha}_i = \psi^{-1}(\bar{\alpha}_i), \widetilde{y}_i = \psi^{-1}(y_i)\}$$

is a basis for TV [6].

Definition 1.6. Let $R_a : GL(n) \to GL(n)$ be a right translation of $GL(n)$ by $a \in GL(n)$ where $R_a(y) = y.a$ for $y \in GL(n)$. Then $B =$

$TR_{a^{-1}}(Y)$ is a tangent vector of $GL(n)$ at its unit element ($e \in GL(n)$), namely $B \in Lie(GL(n))$.

Conversely, for any pair $a \in GL(n)$ and $B \in Lie(GL(n))$, there exists $Y \in T_a(GL(n))$ by $Y = TR_a(B)$. Y can also be written as $Y = [a, B]$ [7].

THEOREM 1.7. *$T(GL(n,\mathbb{R}))$ can be embedded into $GL(2n,\mathbb{R})$ by the following* one-to-one *Lie group homomorphism*

$$\begin{gathered} J_n : T(GL(n,\mathbb{R})) \to GL(2n,\mathbb{R}), \\ J_n([a,B]) = \begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix} \end{gathered} \tag{1.3}$$

for any $a \in GL(n)$ and $B \in Lie(GL(n))$. It can be shown that $J_n([a,0]) = Ta$ [7].

REMARK 1.8. The matrix that corresponds to a linear operator $F \in Aut(V)$ with respect to a fixed basis $\{\alpha_i : 1 \leq i \leq n\} \subset V$ consists of arrays of scalars (F^i_j) determined by

$$F(\alpha_j) = \sum_{i=1}^{n} (F^i_j)\alpha_i \tag{1.4}$$

[2]. Using (1.4), we can define the following group isomorphism

$$Z : GL(n) \to Aut(V) \quad , Z(F^i_j) = F^i_j \alpha_i \otimes \alpha_j \tag{1.5}$$

where $[F^i_j] \in GL(n)$. Another group isomorphism $\check{Z}$ from $GL(2n)$ to $Aut(TV)$ can be defined similar fashion.

DEFINITION 1.9. We define a one-to-one homomorphism

$$\hat{J}_n : T(Aut(V)) \to Aut(TV)$$

with $\hat{J}_n = \check{Z} \circ J_n \circ TZ$. The differential of this homomorphism, which is represented by $(T\hat{J}_n)_{(I,0)} : Lie(T(Aut(V))) \to End(TV)$ where $(I,0)$ refers at unit element, is a Lie algebra homomorphism.

PROPOSITION 1.10. *If X is a left-invariant vector field in a Lie group G, i.e., $TL_aX = X$ for any $a \in G$, then X^v and X^c are left invariant vector fields in the tangent group TG [11]. Moreover if $\{X_1, X_2, ..., X_n\}$ form basis for of the Lie algebra of G. Then their vertical and complete lifts $\{{X_1}^v, {X_2}^v, ..., {X_n}^v, {X_1}^c, {X_2}^c, ..., {X_n}^c\}$ form a basis of the Lie algebra of all left invariant vector fields in TG [11]. If the structure equations are given by*

$$[X_i, X_j] = C_{ij}^k X_k \tag{1.6}$$

C_{ij}^k being the structure constants of G, then we have following equations:

$$\begin{aligned} [X_i, X_j]^c = [{X_i}^c, {X_j}^c] &= C_{ij}^k {X_k}^c \\ [X_i, X_j]^v = [{X_i}^v, {X_j}^c] &= C_{ij}^k {X_k}^v \\ [{X_i}^v, {X_j}^v] &= 0. \end{aligned} \tag{1.7}$$

2 Prolongations of Representations of Lie Algebras

PROPOSITION 2.1. *For all $a, b, c \in G$, $\lambda \in \mathbb{R}$, $X_a, Y_b, Z_c \in T(Lie(G))$, we have following formulas:*

$$\varphi \circ f_a = \sigma_{-1} \circ \varphi \circ \bar{f}_a \tag{2.1}$$
$$\varphi \circ \bar{f}_b = \sigma_{-1} \circ \varphi \circ f_b \tag{2.2}$$
$$\varphi \circ \bar{f}_b \circ \sigma_\lambda = \sigma_\lambda \circ \varphi \circ \bar{f}_b \tag{2.3}$$
$$\varphi \circ f_{\lambda a} = \sigma_\lambda \circ \varphi \circ f_a \tag{2.4}$$
$$\varphi \circ \bar{f}_c \circ \tau_b = \tau_{\varphi(b,c)} \circ \varphi \circ \bar{f}_c \tag{2.5}$$
$$T(\varphi \circ f_a)(Z_c) \oplus T(\varphi \circ f_b)(Z_c) = T(\varphi \circ f_{a+b})(Z_c) \tag{2.6}$$

$$T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)})(X_a)$$

$$= T(\sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ f_c)(X_a) \tag{2.7}$$

$$T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ f_a + \tau_{\varphi(\varphi(a,b),c)} \circ \bar{f}_a \circ \varphi \circ f_b)(Y_b)$$

$$= T(\sigma_{-1} \circ \varphi \circ \bar{f}_{\varphi(c,a)})(Y_b) \tag{2.8}$$

$$T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ f_{\varphi(a,b)} + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ \bar{f}_a \circ \varphi \circ f_b)(Z_c)$$

$$= T(\sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ \bar{f}_a)(Z_c) \tag{2.9}$$

where φ represents the Lie bracket, $T\varphi$ represents the differential of φ and x_i represents coordinate functions of G.

PROOF. For all $x \in Lie(G)$, we have

$$\begin{aligned}(\varphi \circ f_a)(x) &= \varphi(a,x) \\ &= -(\varphi(x,a)) \\ &= \sigma_{-1}((\varphi \circ \bar{f}_a)(x)) \\ &= (\sigma_{-1} \circ \varphi \circ \bar{f}_a)(x).\end{aligned}$$

This proves (2.1). Proofs of (2.2)-(2.5) can be easily shown by the similar way. Therefore, we focus on the rest of the proofs.

Proof of (2.6): Using the coordinate functions of $Lie(G)$, we have

$$(x_i(\tau_{\varphi(b,c)} \circ \varphi \circ f_a) + x_i(\tau_{\varphi(a,c)} \circ \varphi \circ f_b))(X) = x_i(\varphi(b,c) + \varphi(a,X) + \varphi(a,c) + \varphi(b,X))$$

for all $X \in Lie(G)$. Since $\varphi(b,c)$ and $\varphi(a,c)$ are constants, we have

$$\frac{\partial(x_i(\tau_{\varphi(b,c)} \circ \varphi \circ f_a) + x_i(\tau_{\varphi(a,c)} \circ \varphi \circ f_b)}{\partial x_j}|_c = \frac{\partial(x_i \circ \varphi \circ f_{a+b})}{x_j}|_c. \tag{2.10}$$

Using $Z_c = \sum_{j=1}^{m} z_j \frac{\partial}{\partial x_j}|_c$ and (2.10), we have

$$\begin{aligned}
(T(\varphi \circ f_a)(Z_c) \oplus T(\varphi \circ f_b)(Z_c))[x_i] &= (T\tau_{\varphi(b,c)}(T(\varphi \circ f_a)(Z_c)) + T\tau_{\varphi(a,c)}(T(\varphi \circ f_b)(Z_c)))[x_i] \\
&= \sum_{j=1}^{m} z_j \frac{\partial(x_i \circ \tau_{\varphi(b,c)} \circ \varphi \circ f_a + x_i \circ \tau_{\varphi(a,c)} \circ \varphi \circ f_b)}{\partial x_j}|_c \\
&= \sum_{j=1}^{m} z_j \frac{\partial(x_i(\tau_{\varphi(b,c)} \circ \varphi \circ f_a + \tau_{\varphi(a,c)} \circ \varphi \circ f_b))}{\partial x_j}|_c \\
&= \sum_{j=1}^{m} z_j \frac{\partial(x_i \circ \varphi \circ f_{a+b})}{x_j}|_c \\
&= (T(\varphi \circ f_{a+b})(Z_c))[x_i]. \qquad (2.11)
\end{aligned}$$

This completes the proof.

Proof of (2.7): For all $X \in Lie(G)$, we have

$$\begin{aligned}
&(x_i \circ \tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + x_i \circ \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)})(X) \\
&= x_i(\varphi(\varphi(b,c),a) + \varphi(\varphi(X,b),c) + \varphi(\varphi(a,b),c) + \varphi(\varphi(b,c),X)). \quad (2.12)
\end{aligned}$$

Then the differential of 2.12

$$\begin{aligned}
&\frac{\partial x_i(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)})}{\partial x_j} \\
&= \frac{\partial(x_i \circ \sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ f_c)}{\partial x_j}. \qquad (2.13)
\end{aligned}$$

Using (2.12) and (2.13), we finish the proof as follows

$$\begin{aligned}
&T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)})(X_a)[x_i] \\
= \;& X_a[x_i \circ \tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)}] \\
= \;& \sum_{j=1}^{m} X_j \frac{\partial(x_i \circ \tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)})}{\partial x_j}|_a \\
= \;& \sum_{j=1}^{m} X_j \frac{\partial(x_i \circ \sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ f_c)}{\partial x_j}|_a \\
= \;& T(\sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ f_c)(X_a)[x_i].
\end{aligned}$$

Proof of (2.8) and (2.9) can be similarly performed. □

PROPOSITION 2.2. *$(T(Lie(G)), \oplus, \bullet, T\varphi)$ is a Lie algebra.*

PROOF. Using (2.1) and (2.2) for all $(X_a, Y_b) \in T(Lie(G)) \times T(Lie(G))$, we have

$$\begin{aligned} T\varphi(X_a, Y_b) &= T\varphi(T\bar{f}_b(X_a) + Tf_a(Y_b)) \\ &= T(\varphi \circ \bar{f}_b)(X_a) + T(\varphi \circ f_a)(Y_b) \\ &= T(\sigma_{-1} \circ \varphi \circ f_b)(X_a) + T(\sigma_{-1} \circ \varphi \circ \bar{f}_a)(Y_b) \\ &= T\sigma_{-1}(T\varphi(T\bar{f}_a(Y_b) + Tf_b(X_a))) \\ &= T\sigma_{-1}(T\varphi(Y_b, X_a)) \\ &= -1 \bullet T\varphi(Y_b, X_a). \end{aligned} \tag{2.14}$$

(2.14) shows that $T\varphi$ is antisymmetric. For all $X_a, Y_b, Z_c \in T(Lie(G))$ and $\lambda \in \mathbb{R}$, Eqs. (2.3) and (2.4) give

$$\begin{aligned} T\varphi(\lambda \bullet X_a, Y_b) &= T\varphi(T\bar{f}_b(\lambda \bullet X_a) + Tf_{\lambda a})(Y_b) \\ &= T(\varphi \circ \bar{f}_b \circ \sigma_\lambda)(X_a) + T(\varphi \circ f_{\lambda a})(Y_b) \\ &= T(\sigma_\lambda \circ \varphi \circ \bar{f}_b)(X_a) + T(\sigma_\lambda \circ \varphi \circ f_a)(Y_b) \\ &= \lambda \bullet T\varphi(X_a, Y_b). \end{aligned} \tag{2.15}$$

Using Eqs. (2.5) and (2.6), we have

$$\begin{aligned} T\varphi(X_a, Z_c) \oplus T\varphi(Y_b, Z_c) &= T\tau_{\varphi(b,c)}(T\varphi(X_a, Z_c)) + T\tau_{\varphi(a,c)}(T\varphi(Y_b, Z_c)) \\ &= T(\tau_{\varphi(b,c)} \circ \varphi \circ \bar{f}_c)(X_a) + T(\tau_{\varphi(a,c)} \circ \varphi \circ \bar{f}_c)(Y_b) \\ &+ T(\tau_{\varphi(b,c)} \circ \varphi \circ f_a + \tau_{\varphi(a,c)} \circ \varphi \circ f_b)(Z_c) \\ &= T(\varphi \circ \bar{f}_c \circ \tau_b)(X_a) + T(\varphi \circ \bar{f}_c \circ \tau_a)(Y_b) \\ &+ T(\tau_{\varphi(b,c)} \circ \varphi \circ f_a)(Z_c) + T(\tau_{\varphi(a,c)} \circ \varphi \circ f_b)(Z_c) \\ &= T(\varphi \circ \bar{f}_c)((T\tau_b)(X_a) + (T\tau_a)(Y_b)) \\ &+ (T(\varphi \circ f_a)(Z_c) \oplus T(\varphi \circ f_b)(Z_c)) \\ &= T\varphi(T\bar{f}_c(X_a \oplus Y_b) + Tf_{a+b}(Z_c)) \\ &= T\varphi(X_a \oplus Y_b, Z_c). \end{aligned} \tag{2.16}$$

(2.15) and (2.16) show that $T\varphi$ is a bilinear function. Finally, using (2.7), (2.8), and (2.9) we have

$$\begin{aligned}
& T\varphi(T\varphi(X_a, Y_b), Z_c) \oplus T\varphi(T\varphi(Y_b, Z_c), X_a) \\
= & (T(\varphi \circ \bar{f}_c \circ \varphi)(X_a, Y_b) + T(\varphi \circ f_{\varphi(a+b)})(Z_c))) \\
\oplus & (T(\varphi \circ \bar{f}_a \circ \varphi)(Y_b, Z_c) + T(\varphi \circ f_{\varphi(b,c)})(X_a)) \\
= & (T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ \bar{f}_b + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ f_{\varphi(b,c)})(X_a) \\
+ & T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ \bar{f}_c \circ \varphi \circ f_a + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ \bar{f}_a \circ \varphi \circ \bar{f}_c)(Y_b) \\
+ & (T(\tau_{\varphi(\varphi(b,c),a)} \circ \varphi \circ f_{\varphi(a,b)} + \tau_{\varphi(\varphi(a,b),c)} \circ \varphi \circ \bar{f}_a \circ \varphi \circ f_b)(Z_c) \\
= & T(\sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ f_c)(X_a) + T(\sigma_{-1} \circ \varphi \circ \bar{f}_{\varphi(c,a)})(Y_b) \\
+ & T(\sigma_{-1} \circ \varphi \circ \bar{f}_b \circ \varphi \circ \bar{f}_a)(Z_c) \\
= & (T\sigma_{-1} \circ T\varphi)((T\bar{f}_b(T\varphi(Z_c, X_a) + T\bar{f}_{\varphi(c,a)}(Y_b)) \\
= & T\sigma_{-1}(T\varphi(T\varphi(Z_c, X_a), Y_b)) \\
= & -1 \bullet T(\varphi(T\varphi(Z_c, X_a), Y_b)). \qquad (2.17)
\end{aligned}$$

Eq. (2.17) indicates that the Jacobi identity is satisfied. Therefore (2.14), (2.15), (2.16), and (2.17) imply that $(T(Lie(G)), \oplus, \bullet, T\varphi)$ is a Lie algebra. □

LEMMA 2.3. *Let C^k_{ij} be structure constants of G, then we have:*

$$\frac{\partial(x_k \circ \varphi \circ f_{\bar{X}})}{\partial x_j}|_{\bar{Y}} = \bar{a}_i C^k_{ij} \qquad (2.18)$$

$$\frac{\partial(x_k \circ \varphi \circ f_{\bar{Y}})}{\partial x_j}|_{\bar{X}} = \bar{b}_i C^k_{ji} \qquad (2.19)$$

where $\bar{X} = \sum_{i=1}^{m} \bar{a}_i X_i$ *and* $\bar{Y} = \sum_{i=1}^{m} \bar{b}_i X_i$.

PROOF. Using $(x_k \circ \varphi \circ f_{\bar{X}})(X) = \bar{a}_i x_t C^k_{it}$ for all $X \in Lie(G)$, we show

$$\begin{aligned}
\frac{\partial(x_k \circ \varphi \circ f_{\bar{X}})}{\partial x_j}|_{\bar{Y}} &= \frac{\partial(\bar{a}_i x_t C^k_{it})}{\partial x_j} \\
&= \bar{a}_i C^k_{ij}.
\end{aligned}$$

Proof of Eq. (2.19) can be done similarly. □

DEFINITION 2.4. Let $\{X_1, X_2, .., X_m\}$ be a basis for $Lie(G)$ and $\{e_1, e_2, ..., e_m\}$ denote the standard basis of $\mathbb{R}^m$. Then we define a new function

$$\Omega : T(Lie(G)) \to Lie(TG) \tag{2.20}$$

where $\Omega(\bar{X}, \bar{V}) = \sum_{i=1}^{m} \bar{a}_i X_i^c + \bar{v}_i X_i^v$, $\bar{X} = \sum_{i=1}^{m} \bar{a}_i X_i \in Lie(G)$, and $\bar{V} = \sum_{i=1}^{m} \bar{v}_i e_i \in \mathbb{R}^m$.

PROPOSITION 2.5. *$T(Lie(G))$ is algebraically isomorphic to $(Lie(TG), [,])$.*

PROOF. For simplicity, we represent $\{\widetilde{X_i}, \widetilde{y_i}\}$ as a basis for $T(Lie(G))$ where $\widetilde{X_i} = (X_i, 0)$ and $\widetilde{y_i} = (0, e_i)$ [6]. Since $X_i = \sum \delta_j^i X_j$, using (2.20) we have

$$\Omega(\widetilde{X_i}) = \Omega((\sum_{j=1}^{m} \delta_j^i X_j), 0) = \sum_{j}^{m} \delta_j^i X_j^c = X_i^c \tag{2.21}$$

and since $e_i = \sum \delta_j^i e_j$, we have

$$\Omega(\widetilde{y_i}) = \Omega(0, \sum_{j=1}^{m} \delta_j^i e_j) = \sum_{j}^{m} \delta_j^i X_j^v = X_i^v. \tag{2.22}$$

Eqs. (2.21) and (2.22) shows that Ω is a linear isomorphism. Showing that Ω is a Lie algebra homomorphism completes the proof. For that, we take two arbitrary elements $(\bar{X}, \bar{V})$, $(\bar{Y}, \bar{W}) \in T(Lie(G))$ where $\bar{X} = \sum_{i=1}^{m} a_i X_i$,

$\bar{Y} = \sum_{i=1}^{m} b_i X_i$ and $\bar{V} = \sum_{i=1}^{m} v_i e_i$, $\bar{W} = \sum_{i=1}^{m} w_i e_i$. Then, we have

$$\begin{aligned} T\varphi((\bar{X},\bar{V}),(\bar{Y},\bar{W}))[x_k] &= (\bar{Y},\bar{W})[x_k \circ \varphi \circ f_{\bar{X}}] + (\bar{X},\bar{V})[x_k \circ \varphi \circ \bar{f}_{\bar{Y}}] \\ &= \sum_{i,j=1}^{m} (\bar{w}_i \frac{\partial (x_k \circ \varphi \circ f_{\bar{X}})}{\partial x_j}|_{\bar{Y}} + \bar{v}_i \frac{\partial (x_k \circ \varphi \circ \bar{f}_{\bar{Y}})}{\partial x_j}|_{\bar{X}}) \\ &= \sum_{i,j=1}^{m} (\bar{w}_j \bar{a}_i C_{ij}^k + \bar{v}_j \bar{b}_i C_{ji}^k) \\ &= \sum_{i,j=1}^{m} (\bar{w}_j \bar{a}_i - \bar{v}_j \bar{b}_i) C_{ij}^k. \end{aligned} \tag{2.23}$$

(2.23) implies that

$$T\varphi((\bar{X},\bar{V}),(\bar{Y},\bar{W})) = (\sum \bar{a}_i \bar{b}_j [X_i, X_j], \sum (\bar{w}_j \bar{a}_i - \bar{v}_j \bar{b}_i) C_{ij}^k e_k). \tag{2.24}$$

Using Eqs. (1.7) and (2.24), we obtain

$$\begin{aligned} \Omega(T\varphi((\bar{X},\bar{V}),(\bar{Y},\bar{W}))) &= \sum \bar{a}_i \bar{b}_j C_{ij}^k X_k^c + \sum (\bar{w}_j \bar{a}_i - \bar{v}_j \bar{b}_i) C_{ij}^k X_k^v \\ &= \sum \bar{a}_i \bar{b}_j [X_i, X_j]^c + \sum (-\bar{w}_j \bar{a}_i + \bar{v}_j \bar{b}_i)[X_j, X_i]^v \\ &= \sum \bar{a}_i \bar{b}_j [X_i, X_j]^c + \sum (-\bar{w}_j \bar{a}_i + \bar{v}_j \bar{b}_i)[X_j, X_i]^v \\ &= [\Omega(\bar{X},\bar{V}), \Omega(\bar{Y},\bar{W})]. \end{aligned} \tag{2.25}$$

Eq. (2.25) shows that Ω is a Lie algebra homomorphism, therefore this finishes the proof. □

PROPOSITION 2.6. *Let (g,φ) and (h,γ) be two Lie algebras and $F : g \to h$ be a Lie algebra homomorphism. Then TF is a Lie algebra homomorphism.*

PROOF. Since F is a Lie algebra homomorphism, it is a linear function. By definition (1.3), TF is a linear function. Showing that TF preserves Lie brackets, we complete the proof.

Since F is a Lie algebra homomorphism, $F(\varphi(X,Y)) = \gamma(F(X), F(Y))$ for all $X, Y \in g$. This means

$$F \circ \varphi = \gamma \circ (F \times F). \tag{2.26}$$

From (2.26), we can write $TF \circ T\varphi = T\gamma \circ (TF \times TF)$. This leads to

$$\begin{aligned} TF(T\varphi(\bar{X}, \bar{Y})) &= (TF \circ T\varphi)(\bar{X}, \bar{Y}) \\ &= T\gamma \circ (TF \times TF)(\bar{X}, \bar{Y}) \\ &= T\gamma(TF(\bar{X}), TF(\bar{Y})) \end{aligned} \tag{2.27}$$

where $\bar{X} \in Tg$ and $\bar{Y} \in Tg$. Therefore (2.27) implies that TF preserves Lie brackets. □

PROPOSITION 2.7. *Let $\phi : Lie(G) \to End(V)$ be a Lie algebra representation, $\Omega : T(Lie(G)) \to Lie(TG)$ and $\Omega' : T(End(V)) \to Lie(Oto(TV))$ be Lie algebra isomorphisms which are defined by Eq. (2.4). Then $\widetilde{\phi} : Lie(TG) \to End(TV)$ is a representation of $Lie(TG)$ defined as*

$$\widetilde{\phi} = (T\hat{J}_n)_{(I,0)} \circ \Omega' \circ T\phi \circ \Omega^{-1} \tag{2.28}$$

where $\hat{J}_n$ represents a one-to-one Lie group homomorphism from $T(Aut(V))$ to $Aut(TV)$ and $T(\hat{J}_n)_{(I,0)}$ represents the differential of $\hat{J}_n$ at $(I, 0)$ (refer to Definition 1.9).

PROOF. By Proposition (2.7), $T\phi$ is a Lie algebra homomorphism. Since Ω and Ω' are Lie algebra isomorphisms and $(T\hat{J}_n)_{(I,0)}$ is a Lie algebra homomorphism then $\widetilde{\phi}$ is a Lie algebra homomorphism , i.e., $\widetilde{\phi}$ is a representation of $Lie(TG)$. □

DEFINITION 2.8. The Lie algebra representation $\widetilde{\phi}$ given in (2.28) is referred to as the prolongation of ϕ.

3 Conclusion

In this paper, we have studied prolongations of real representations of Lie algebras. In particular, we have obtained representations of Lie algebras of tangent bundles of Lie groups. First, we have shown that a tangent bundle of a given Lie algebra attains a Lie algebra structure. Then, we have proven that this tangent bundle is algebraically isomorphic to a Lie algebra of the tangent bundle of corresponding Lie group. Using these, we have defined prolongations of representations of Lie algebras. In future, we will study the relationships between the prolongations studied here and the prolongations presented in [6].

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DIRAC CONSISTENCY OF THE ALGEBRA OF HAMILTONIAN CONSTRAINTS IN REDUCED 4-D GENERAL RELATIVITY

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Abstract

In this paper we provide an action related to a certain sector of general relativity where the algebra of Hamiltonian constraints forms a first class system. This action is a Dirac-consistent stand-alone action with two physical degrees of freedom per point. In this paper we provide the steps necessary to transform this new action to and from the associated sectors of the Ashtekar theory and a certain antecedent of the pure spin connection formulation by Capovilla, Dell and Jacobson.

1 Introduction

The invariance of Einstein's theory of general relativity (GR) under general coordinate transformations is explicit at the covariant level of the theory, where space and time appear on equal footing. In a canonical treatment one formulates the theory using variables defined on 3-dimensional spatial hypersurfaces which evolve in time. This almost inevitably introduces a 3+1 splitting of the theory, and one must verify in the end that the original invariance has been preserved under this splitting. In this paper we will probe this principle using a theory related to GR, specifically within the realm of time reparametrizations. Let us consider a general transformation of coordinates $x \in M$, where M is a 4-dimensional spacetime manifold

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}(x). \tag{1}$$

As shown in [1], the form variation of a field $F(x)$, $\delta_0 F(x) = F'(x) - F(x)$ should be clearly distinguished from its total variation, $\delta F(x) \equiv F'(x') - F(x)$. Form variation and differentiation are commuting operations, and when $x' - x$ is infinitesimally small, we get

$$\delta F(x) \sim \delta_0 F(x) + \xi^{\mu}\partial_{\mu} F(x). \tag{2}$$

A scalar field $\varphi(x) \in M$ is a field which is invariant with respect to transformations (2): $\varphi'(x') = \varphi(x)$. As a consequence, the form variation of φ is given by transformation law [1]

$$\delta_0\varphi(x) = -\xi^{\mu}\partial_{\mu}\varphi(x). \tag{3}$$

Equation (1) as an infinitesimal general coordinate transformation defines the following vector field $\xi = \xi^{\mu}\partial_{\mu} \in M$ as realized in (2). The commutator of any two vector fields $\xi, \zeta \in M$ is given by the Lie bracket

$$[\xi^{\mu}\partial_{\mu}, \zeta^{\nu}\partial_{\nu}] = (\xi^{\mu}\partial_{\mu}\zeta^{\nu} - \zeta^{\mu}\partial_{\mu}\xi^{\nu})\partial_{\nu}, \tag{4}$$

which defines a Lie algebra of general coordinate transformations.

To approach the question of whether there exists a formulation of GR where the Lie algebra (4) can be realized at the canonical level, let us perform a 3+1 splitting of (4) into purely spatial and temporal vector fields $\xi^{\mu} = (0, N^1, N^2, N^3)$ and $\xi^{\mu} = (N, 0, 0, 0)$ for comparison. This yields the following algebra

$$\begin{aligned} [N^i\partial_i, N^j\partial_j] &= (M^i\partial_i N^j - N^i\partial_i M^j)\partial_j; \\ [N^i\partial_i, N\partial_0] &= (N^i\partial_i N)\partial_0 - (N\dot{N}^i)\partial_i \\ [M\partial_0, N\partial_0] &= (M\dot{N} - N\dot{M})\partial_0. \end{aligned} \tag{5}$$

The Poisson algebra of hypersurface deformations for general relativity has been computed by Teitelboim [2]

$$\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\}=H_k\big[N^i\partial^k M_i-M^i\partial^k N_i\big];$$
$$\{H(N),\vec{H}[\vec{N}]\}=H[N^i\partial_i N]$$
$$\{H(N),H(M)\}=H_i\big[\big(N\partial_j M-M\partial_j N\big)q^{ij}\big],\tag{6}$$

where $H_\mu=(H,H_i)$ are the Hamiltonian and diffeomorphism constraints, and q^{ij} are phase space dependent structure functions. If one could make the identifications

$$H_\mu\sim\partial_\mu\longrightarrow H\sim\partial_0;\quad H_i\sim\partial_i,\tag{7}$$

then there would be an isomorphism between (5) and (6) with respect to purely spatial diffeomorphisms, which form a subalgebra of (4). However, equation (5) states that temporal diffeomorphisms should also form a subalgebra of (4), which clearly is not the case in (6).

A direct implication of (6) is the nonexistence of a canonical formulation of GR, in the full theory, which evolves purely under the dynamics of the Hamiltonian constraint.[1] In this paper we will propose an action I_{Kin}, which is directly related to a certain restricted sector of GR in a sense which we will make precise. We will show that the temporal part of the algebra (5) is realized via Poisson brackets on I_{Kin}, which is the main result of this paper. The question of whether I_{Kin} is equivalent or not to GR is one which we will not address in this paper. Rather, we will show that I_{Kin} is a theory with 2 degrees of freedom per point on its reduced phase space, which is directly transformable into certain subspaces of the original theory of GR in ways which we will clearly demonstrate.

The title of this paper refers to a 'reduced' 4-dimensional GR theory, which is presented as I_{Kin}. We will like to clarify that we have not shown in this paper that I_{Kin} follows from full GR in the sense of a reduced phase space procedure, which involves solving constraints and gauge-fixing. Rather, we will present I_{Kin} as a restriction by hand from full GR to a certain subspace upon which our analysis will be carried out. It will become clear that the action I_{Kin} is still a stand-alone action irrespective of the issue of its precise relation GR.[2] The organization of this paper is as follows. Sections 2 and 3 present the action I_{Kin} as the starting point, which is a totally constrained system with a single constraint which we have named a

[1] Another way to state this is that the Hamiltonian constraint H does not form a first class system. This is because the Poisson bracket of two Hamiltonian constraints yields a diffeomorphism constraint. So (6) suggests that to be consistent, the diffeomorphism constraint H_i must be part of the theory in addition to the Hamiltonian constraint H.

[2] So while we will not claim here that I_{Kin} is the actual reduced phase space for gravity, we will present it as motivation for the prospect that such a formulation for GR, where this or something similar might perhaps be realizable, cannot be ruled out.

Hamiltonian constraint. We carry out the Dirac procedure for constrained systems, showing that I_{Kin} is Dirac consistent at the classical level, and with a physical phase space having two degrees of freedom per point. Sections 4 and 5 present the transformations which take I_{Kin} to and from certain sectors of GR, specifically the restriction of full general relativity to the diagonal subspace of the Ashtekar and other variables and with no Gauss' law and diffeomorphism constraint. Our main result will be to show that this is still consistent, even if it turns out to be the case that I_{Kin} is not equivalent to GR. Section 6 is a short discussion and conclusion of our results. In Appendix A, we derive the set of configurations exhibiting the same features as I_{Kin} as we have presented in this paper.

On a final note regarding index conventions for this paper, lowercase symbols $a, b, c, \ldots$ from the beginning part of the Latin alphabet signify internal SO(3,C) indices, while those from the middle $i, j, k, \ldots$ are spatial indices. Both sets of indices will take values 1, 2 and 3. Greeek indices μ, ν will denote spacetime indices, which take values 0, 1, 2 and 3.

2 The starting action

Consider the phase space $\Omega_{Kin} = (\Gamma_{Kin}, P_{Kin})$ of a system with configuration and momentum space variables $\Gamma_{Kin} = (X, Y, T)$ and $P_{Kin} = (\Pi_1, \Pi_2, \Pi)$ defined on a 4-dimensional spacetime manifold of topology $M = \Sigma \times R$, where Σ is a 3-dimensional spatial hypersurface with R as the time direction. The variables are in general complex, and the configuration space variables take on the ranges $-\infty < |X|, |Y|, |T| < \infty$. The following mass dimensions have been assigned to the variables

$$[\Pi_1] = [\Pi_2] = [\Pi] = 1; \quad [X] = [Y] = [T] = 0. \tag{8}$$

From these variables can be constructed the following kinematic phase space action for a totally constrained system

$$I = -\frac{i}{G}\int dt \int_\Sigma d^3x(\Pi_1\dot{X} + \Pi_2\dot{Y} + \Pi\dot{T}) - iH[N], \tag{9}$$

where G is Newton's gravitational constant. The function H is smeared by an auxilliary field N, forming a Hamiltonian density $H[N]$ given by

$$H[N] = \int_\Sigma d^3xNUe^{-T/2}\Phi \tag{10}$$

where the quantities in (10) are defined as follows. First we have Φ, given by

$$\Phi = \sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big[\Big(k + e^T\Big(\frac{1}{\Pi} + \frac{1}{\Pi+\Pi_1} + \frac{1}{\Pi+\Pi_2}\Big)\Big] \tag{11}$$

where $k = \frac{\Lambda}{a_0^3}$ is a numerical constant.[3] There are no spatial derivatives in any of the quantities in (11), and all spatial derivatives in the theory (9) are confined to the quantity U in (10), given by

$$U = \Big[1 + e^{-T}\big((\partial_2 Z)(\partial_3 X)(\partial_1 Y) - (\partial_3 Y)(\partial_1 Z)(\partial_2 X)\big)$$
$$+e^{-2X}(\partial_1 Y)(\partial_1 Z) + e^{-2Y}(\partial_2 Z)(\partial_2 X) + e^{-2Z}(\partial_3 X)(\partial_3 Y)\Big]^{1/2} \quad (12)$$

with $Z = T - X - Y$. We have defined

$$\partial_1 = \frac{\partial}{\partial y^1}; \quad \partial_2 = \frac{\partial}{\partial y^2}; \quad \partial_3 = \frac{\partial}{\partial y^3}, \quad (13)$$

where y^1, y^1 and y^3 are dimensionless spatial coordinates in 3-space Σ.

The canonical structure of (9) yields the following Poisson brackets amongst fundamental phase space variables

$$\{X(x,t),\Pi_1(y,t)\} = \{Y(x,t),\Pi_2(y,t)\} = \{T(x,t),\Pi(y,t)\} = -iG\delta^{(3)}(x,y), \quad (14)$$

with all other brackets vanishing. Note that this induces the following canonical Poisson bracket between any two phase space function $f, g \in C^\infty(\Omega_{Kin})$

$$\{f,g\} = \int_\Sigma d^3x\Big[\frac{\delta f}{\delta \Pi_1}\frac{\delta g}{\delta X} - \frac{\delta g}{\delta \Pi_1}\frac{\delta f}{\delta X} + \frac{\delta f}{\delta \Pi_2}\frac{\delta g}{\delta Y} - \frac{\delta g}{\delta \Pi_2}\frac{\delta f}{\delta Y} + \frac{\delta f}{\delta \Pi}\frac{\delta g}{\delta T} - \frac{\delta g}{\delta \Pi}\frac{\delta f}{\delta T}\Big]. \quad (15)$$

Since the Hamiltonian of (9) consists purely of a constraint proportional to Φ, then it is appropriate to proceed with the Dirac analysis for totally constrained systems [3].

The velocity $\dot{N}$ does not appear in the starting action (9), which implies as a primary constraint the vanishing of its conjugate momentum Π_N

$$\Pi_N = \frac{\delta I}{\delta \dot{N}} = 0. \quad (16)$$

As a consistency condition we must require that Π_N be preserved in time, which leads to the secondary constraint

$$\dot{\Pi}_N = \frac{\delta I}{\delta N} = H = Ue^{-T/2}\Phi = 0. \quad (17)$$

We must then check for the preservation of (17) in time, which is the same as checking for closure of the algebra of Hamiltonian (10) under Poisson brackets (15).

[3]We will identify Λ as the cosmological constant, and a_0 is a numerical constant of mass dimension $[a_0] = 1$.

3 Poisson algebra of the Hamiltonian constraint

We will now compute the Poisson algebra of two Hamiltonians. There exist phase space functions $q^I = q^I(\Omega_{Kin})$ such that the functional derivatives of (10) with respect to momentum space variables are weakly of the form

$$\frac{\delta H[N]}{\delta \Pi_1} \sim Nq^1; \quad \frac{\delta H[N]}{\delta \Pi_2} \sim Nq^2; \quad \frac{\delta H[N]}{\delta \Pi} \sim Nq^3 \tag{18}$$

where we have defined

$$q^1 = -Ue^{T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big(\frac{1}{\Pi+\Pi_1}\Big)^2$$
$$q^2 = -Ue^{T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big(\frac{1}{\Pi+\Pi_2}\Big)^2;$$
$$q^3 = -Ue^{T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big[\Big(\frac{1}{\Pi}\Big)^2 + \Big(\frac{1}{\Pi+\Pi_1}\Big)^2 + \Big(\frac{1}{\Pi+\Pi_2}\Big)^2\Big]. \tag{19}$$

For the configuration space the relevant contributions will arise from integration of U by parts, which transfers the spatial gradients away from the variables whose functional derivatives are being evaluated. For functional derivatives with respect to the 'coordinate' X we have that

$$-\frac{\delta H[M]}{\delta X} = \partial_i(\eta^i_1 M\Phi) + \frac{1}{U}Me^{T/2}\Phi\Big(-e^{-2X}(\partial_1 Y)(\partial_1 Z) + e^{-2Z}(\partial_3 X)(\partial_3 Y)\Big), \tag{20}$$

where the following quantities have been defined

$$\eta^1_1 = \frac{1}{2U}e^{T/2}\Big(-e^{-2X}(\partial_1 Y) + e^{-T}(\partial_2 X)(\partial_3 Y)\Big);$$
$$\eta^2_1 = \frac{1}{2U}e^{T/2}\Big(e^{-2Y}\partial_2(Z-X) - e^{-T}\big((\partial_3 X)(\partial_1 Y) + (\partial_3 Y)(\partial_1 Z)\big)\Big)$$
$$\eta^3_1 = \frac{1}{2U}e^{T/2}\Big(e^{-2Z}(\partial_3 Y) + e^{-T}(\partial_1 Y)(\partial_2 Z)\Big). \tag{21}$$

For functional derivatives with respect to the 'coordinate' Y we have

$$-\frac{\delta H[M]}{\delta Y} = \partial_i(\eta^i_1 M\Phi) + \frac{1}{U}Me^{T/2}\Phi\Big(-e^{-2Y}(\partial_2 Z)(\partial_2 X) + e^{-2Z}(\partial_3 X)(\partial_3 Y)\Big), \tag{22}$$

where the following quantities have been defined

$$\eta^1_2 = \frac{1}{2U}e^{T/2}\Big(e^{-2X}\partial_1(Z-Y) + e^{-T}\big((\partial_2 Z)(\partial_3 X) + (\partial_2 X)(\partial_3 Y)\big)\Big);$$
$$\eta^2_2 = \frac{1}{2U}e^{T/2}\Big(-e^{-2Y}\big((\partial_2 X) + (\partial_3 X)(\partial_1 Y)\big)\Big);$$
$$\eta^3_2 = \frac{1}{2U}e^{T/2}\Big(e^{-2Z}(\partial_3 X) - e^{-T}(\partial_1 Z)(\partial_2 X)\Big). \tag{23}$$

For functional derivatives with respect to the 'coordinate' T we have

$$-\frac{\delta H[M]}{\delta T} = \partial_i(\eta_3^i M\Phi) - \frac{1}{2U}Me^{T/2}\Big(e^{-T}\big((\partial_2 Z)(\partial_3 X)(\partial_1 Y) - (\partial_3 Y)(\partial_1 Z)(\partial_2 X)\big) - e^{-2Z}(\partial_3 X)(\partial_3 Y), \tag{24}$$

where the following quantities have been defined

$$\eta_3^1 = \frac{1}{2U}\Big(e^{-2X}(\partial_1 Y) - e^{-T}(\partial_2 X)(\partial_3 Y)\Big);$$
$$\eta_3^2 = \frac{1}{2U}\Big(e^{-2Y}(\partial_2 X) + e^{-T}(\partial_3 X)(\partial_1 Y)\Big);$$
$$\eta_3^3 = 0. \tag{25}$$

Let us now compute the individual terms contributing to the Poisson brackets between two Hamiltonian constraints smeared by auxilliary fields N and M. Using (18) and (20), (22) and (24) for the contribution due to (Π_1, X) we have

$$\int_\Sigma d^3x\Big(\frac{\delta H[N]}{\delta \Pi_1(x)}\frac{\delta H[M]}{\delta X(x)} - \frac{\delta H[M]}{\delta \Pi_1(x)}\frac{\delta H[N]}{\delta X(x)}\Big)$$
$$= \int_\Sigma d^3x\Big((Nq^1)\partial_i(\eta_1^i M\Phi) - (Mq^1)\partial_i(\eta_1^i N\Phi)\Big) = \int_\Sigma d^3x q^1\eta_1^i(N\partial_i M - M\partial_i N)\Phi. \tag{26}$$

Due to antisymmetry with respect to the difference of scalar functions, the only nontrivial contributions to (26) are from spatial derivatives acting on the functions M and N. Similarly for the (Π_2, Y) contribution we have

$$\int_\Sigma d^3x\Big(\frac{\delta H[N]}{\delta \Pi_2(x)}\frac{\delta H[M]}{\delta Y(x)} - \frac{\delta H[M]}{\delta \Pi_2(x)}\frac{\delta H[N]}{\delta Y(x)}\Big) = \int_\Sigma d^3x q^2\eta_2^i(N\partial_i M - M\partial_i N)\Phi. \tag{27}$$

For the contribution to Poisson brackets due to (Π, T) we have

$$\int_\Sigma d^3x\Big(\frac{\delta H[N]}{\delta \Pi(x)}\frac{\delta H[M]}{\delta T(x)} - \frac{\delta H[M]}{\delta \Pi(x)}\frac{\delta H[N]}{\delta T(x)}\Big)$$
$$= \int_\Sigma d^3x\Big((Nq^3)\big(\partial_i(\eta_3^i M\Phi) + MC\big) - (Mq^1)\big(\partial_i(\eta_3^i N\Phi) + NC\big)\Big)$$
$$= \int_\Sigma d^3x\Big((Nq^3)\partial_i(\eta_3^i M\Phi) - (Mq^3)\partial_i(\eta_3^i N\Phi)\Big) + \int_\Sigma d^3x\Big[(Nq^3)MC - (Mq^3)NC\Big]. \tag{28}$$

The second integral on the last line on the right hand side of (28) vanishes, and the first integral simplifies to

$$\int_\Sigma d^3x q^3\eta_3^i(N\partial_i M - M\partial_i N)\Phi. \tag{29}$$

Combining the results of (29), (27) and (26), we have that

$$\{H[N], H[M]\} = \int_\Sigma d^3x q^I\eta_I^j(N\partial_i M - M\partial_i N)\Phi = H[N, M], \tag{30}$$

namely that the Poisson bracket of two Hamiltonian constraints is a Hamiltonian constraint with phase space dependent structure functions. The result is that the classical Hamiltonian constraints algebra for (9) closes with no further constraints on the system.

The classical constraints algebra of (9) closes, which implies that Ω_{Kin} constitutes a first class system. A degree-of-freedom counting yields

$$3\ (momentum) + 3\ (config.) - 1\ (First\ Class\ Constraint)$$
$$-1\ (Gauge - fixing) = 4\ phase\ space\ D.O.F., \tag{31}$$

which corresponds to two physical degrees of freedom per point. The first class constraint is the Hamiltonian constraint H, and gauge-fixing of I_{Kin} to its physical degrees of freedom involves factoring out the gauge orbits generated by H in conjunction with making a choice of the auxilliary field N. With two propagating degrees of freedom on its physical phase space, then we know that (9) is not a topological field theory.

4 Relation of I_{Kin} to general relativity

There are at least two ways in which the starting action (9) is related to general relativity, which we will explain in the remainder of this paper. (i) The first is the relation of I_{Kin} to gravity in the Ashtekar variables (See e.g. [4], [5] and [6]). The Ashtekar action is given by

$$I_{Ash} = \int dt \int_\Sigma d^3x \Big[\widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a$$
$$-\epsilon_{ijk} N^i \widetilde{\sigma}^j_a B^k_a - \frac{i}{2} \underline{N} \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \Big(\frac{\Lambda}{3} \widetilde{\sigma}^k_c + B^k_c\Big)\Big], \tag{32}$$

where $\widetilde{\sigma}^i_a$ is the densitized triad with $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ the densitized lapse function. The configuration space variable A^a_i is a gauge connection valued in SO(3,C).The fields N^i and A^a_0 in (32) are auxilliary fields smearing the Gauss' law and the diffeomorphism constraints. Note that the constraints algebra of two Hamiltonian constraints from (32) is given by [4]

$$\{H[M], H[N]\} = H_i[q^{ij}(M\partial_i N - N\partial_i M)], \tag{33}$$

which has the same form as (6). We will come back to this point later in this paper.

(ii) The second way is the relation of I_{Kin} to a certain action appearing in [7], which forms an intermediate step in obtaining the pure spin connection formulation I_{CDJ} from Plebanski's theory of gravity [8]. This action is

$$I_{(2)} = -\frac{i}{G} \int dt \int_\Sigma d^3x \Big[\frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - i\eta\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big], \tag{34}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$ is the curvature of a 4-dimensional $SO(3, C)$ connection A^a_μ, and η is a scalar density. We would like to clarify that (34) is not the final action proposed by Capovilla, Dell and Jacobson in [7]. The proposed action I_{CDJ}, which we will not display here, was obtained by elimination of the field Ψ_{ae} from (34), which we will refer to in this paper as the 'CDJ action antecedent'. We will now show that I_{Kin} can be seen as a restriction of (32) in conjunction with (34) to certain sectors of phase space.

4.1 Relation of I_{Kin} to the CDJ action antecedent

Consider the following transformations

$$\Pi = a_0^3 e^T \lambda_3; \quad \Pi + \Pi_1 = a_0^3 e^T \lambda_1; \quad \Pi + \Pi_2 = a_0^3 e^T \lambda_2 \tag{35}$$

for the momentum space variables P_{Kin}, and

$$X = \ln\Bigl(\frac{a_1}{a_0}\Bigr); \quad Y = \ln\Bigl(\frac{a_2}{a_0}\Bigr); \quad T = \ln\Bigl(\frac{a_1 a_2 a_3}{a_0^2}\Bigr) \tag{36}$$

for the configuration space variables Γ_{Kin}, where a_0 is a numerical constant of mass dimension $[a_0] = 1$. Note that the new coordinates have the ranges $0 < |a_a| < \infty$ for $a = 1, 2, 3$, which forms a 3-dimensional functional manifold per point with the origin $a_a = 0$ missing. Let us also make the definitions

$$x^1 = \frac{y^1}{a_0}; \quad x^2 = \frac{y^2}{a_0}; \quad x^3 = \frac{y^3}{a_0} \tag{37}$$

with y^1, y^2 and y^3 the dimensionless spatial coordinates in (13), whence $[x^i] = -1$. Substitution of (36) and (37) into (12) yields

$$\begin{aligned} U = (a_1 a_2 a_3)^{-1} \Bigl[& (a_1 a_2 a_3)^2 + (\partial_2 a_3)(\partial_3 a_1)(\partial_1 a_2) \\ & -(\partial_3 a_2)(\partial_1 a_3)(\partial_2 a_1) + a_2 a_3 (\partial_1 a_2)(\partial_1 a_3) + a_3 a_1 (\partial_2 a_3)(\partial_2 a_1) \\ & + a_1 a_2 (\partial_3 a_1)(\partial_3 a_2) \Bigr]^{1/2} = (\det A)^{-1} (\det B)^{1/2}, \end{aligned} \tag{38}$$

from which one recognizes U as the square root of the determinant of the magnetic field B^i_a for a diagonal connection $A^a_i = diag(a_1, a_2, a_3)$, with the leading order term in $(\det A)$ factored out. In matrix form this is given by

$$a^a_i = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}; \quad b^i_a = \begin{pmatrix} a_2 a_3 & -\partial_3 a_2 & \partial_2 a_3 \\ \partial_3 a_1 & a_3 a_1 & -\partial_1 a_3 \\ -\partial_2 a_1 & \partial_1 a_2 & a_1 a_2 \end{pmatrix}.$$

Substitution of (35), (36) and(38) into (9) yields

$$I=-\frac{i}{G}\int dt\int_{\Sigma}d^3x\Big(\lambda_1a_2a_3\dot{a}_1+\lambda_2a_3a_1\dot{a}_2+\lambda_3a_1a_2\dot{a}_3$$
$$-iN(\mathrm{det}b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\Big).\qquad(39)$$

Equation (39) is nothing other than the 3+1 decomposition of (34) with the Gauss' law constraint missing, with a phase space restricted to diagonal variables $A^a_i=diag(a_1,a_2,a_3)$ and $\Psi_{ae}=diag(\lambda_1,\lambda_2,\lambda_3)$. Equation (39) can be seen as the result of choosing $A^a_0=0$ at the level of the action (34), which in certain interpretations corresponds to a gauge-fixing choice. In this sense the possibility exists that (39), while shown under the guise of (9) to be a Dirac consistent theory, could conceivably be a different theory from (34) in actuality.

The action (39) has the peculiar feature that its canonical one form does not have any spatial derivatives. But there are spatial derivatives contained in the factor $(\mathrm{det}b)^{1/2}$ in its Hamiltonian, and therefore (39) is not a minisuperspace theory. The canonical one-form in (39) can be seen as the restriction to diagonal variables of the object

$$\boldsymbol{\theta}=\int_{\Sigma}d^3x\Psi_{ae}B^i_e\dot{A}^a_i\Big|_{diag(\Psi;A)}.\qquad(40)$$

It so happens, since all spatial derivatives from the magnetic field B^i_a occur in the off-diagonal matrix positions when $\dot{A}^a_i$ is diagonal, that the contraction with a diagonal matrix $\Psi_{ae}=diag(\lambda_1,\lambda_2,\lambda_3)$ annihilates these derivative terms. There are six distinct configurations of A^a_i which exhibit this feature, and we will refer to these configurations as 'quantizable configurations' of configuration space Γ_q. The configurations Γ_q are given by

$$a^a_i=\begin{pmatrix}a^1_1&0&0\\0&a^2_2&0\\0&0&a^3_3\end{pmatrix},\begin{pmatrix}a^1_1&0&0\\0&0&a^2_3\\0&a^3_2&0\end{pmatrix},\begin{pmatrix}0&a^2_1&0\\a^1_2&0&0\\0&0&a^3_3\end{pmatrix},$$
$$\begin{pmatrix}0&a^2_1&0\\0&0&a^3_2\\a^1_3&0&0\end{pmatrix},\begin{pmatrix}0&0&a^3_1\\a^1_2&0&0\\0&a^2_3&0\end{pmatrix},\begin{pmatrix}0&0&a^3_1\\0&a^2_2&0\\a^1_3&0&0\end{pmatrix}\in\Gamma_q,$$

namely the set of connections a^a_i having three nonvanishing elements, and with $\mathrm{det}a\neq0$. The proof of this is provided in Appendix A. Note that the same Dirac procedure as in sections 2 and 3 can be applied to each of the six configurations Γ_q just as for the diagonal one considered. Hence there are six separate sectors of a theory of I_{Kin} which can be studied.

5 Relation of I_{Kin} to the Ashtekar variables

To see the relation of (9) to the Ashtekar variables, let us perform a canonical analysis at the level of (39). The momenta canonically conjugate to the (diagonal) connection are given by $p_a = \delta I_{Kin}/\delta\dot{a}_a$, namely

$$p_1 = \lambda_1 a_2 a_3; \quad p_2 = \lambda_2 a_3 a_1; \quad p_3 = \lambda_3 a_1 a_2. \tag{41}$$

Let us now substitute (41) into the Hamiltonian density of (39). This yields

$$\begin{aligned} H &= (\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big) \\ &= (\det b)^{1/2}\frac{\sqrt{p_1p_2p_3}}{(a_1a_2a_3)}\Big(\Lambda + \frac{a_2a_3}{p_1} + \frac{a_3a_1}{p_2} + \frac{a_1a_2}{p_3}\Big) \\ &= U(p_1p_2p_3)^{-1/2}\big(\Lambda p_1p_2p_3 + p_1p_2(a_1a_2) + p_2p_3(a_2a_3) + p_3p_1(a_3a_1)\big), \end{aligned} \tag{42}$$

with U given by (38). Substitution of (41) back into (39) yields the action

$$I[p,a] = \int dt \int_\Sigma d^3x p_a \dot{a}^a - iNU(\det p)^{-1/2}H, \tag{43}$$

with U as defined as in (38) and with

$$H = \Lambda p_1p_2p_3 + p_1p_2(a_1a_2) + p_2p_3(a_2a_3) + p_3p_1(a_3a_1). \tag{44}$$

In the case where the connection A^a_i is spatially homogeneous, all derivatives in U vanish and (43) reduces to a diagonal Bianchi I model. But $a_a = a_a(x)$ in general contains three degrees of freedom per point, corresponding to three free functions of position and time. The spatial derivatives $\partial_i a$ in general are nonzero, and therefore (43), as well as (39), are not minisuperspace theories.

The action (32) with the Gauss' law and diffeomorphism constraints removed by hand is given by[4]

$$I = \int dt \int_\Sigma d^3x \Big[\widetilde{\sigma}^i_a \dot{A}^a_i - \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\Big(\frac{\Lambda}{3}\widetilde{\sigma}^k_c + B^k_c\Big)\Big]. \tag{45}$$

Recall that the Poisson bracket between two Hamiltonian constraints is a diffeomorphism constraint as in (33). Since there is no diffeomorphism constraint contained in (45), then this action in its present form cannot be Dirac consistent in the full theory. But suppose that we restrict (45) to the subspace of spatially inhomogeneous diagonal variables

$$\widetilde{\sigma}^i_a = \begin{pmatrix} p_1(x) & 0 & 0 \\ 0 & p_2(x) & 0 \\ 0 & 0 & p_3(x) \end{pmatrix}; \quad A^a_i = \begin{pmatrix} a_1(x) & 0 & 0 \\ 0 & a_2(x) & 0 \\ 0 & 0 & a_3(x) \end{pmatrix}$$

[4] The removal of Gauss' law and the diffeomorphism constraints by hand can in certain interpretations be seen as a gauge-fixing choice $N^i = A^a_0 = 0$ at the level of the action (32). This implies in certain interpretations that (45) and (32) most likely are two inequivalent theories.

with 3 D.O.F. per point. Then for $\widetilde{\sigma}^i_a = \delta^i_a p_a$ and $A^a_i = \delta^a_i a_a$ with no summation over a, the action (45) is given by

$$I = \int dt \int_\Sigma d^3x \Big[\widetilde{\sigma}^i_a \dot{A}^a_i - iN(\det\widetilde{\sigma})(\Lambda + (\widetilde{\sigma}^{-1})^a_i B^i_a \Big] \Big|_{Diag(A;\widetilde{\sigma})}$$

$$= \int dt \int_\Sigma d^3x \Big[p_a \dot{a}_a - i\underline{N}(\Lambda p_1 p_2 p_3 + p_1 p_2 (a_1 a_2) + p_2 p_3 (a_2 a_3) + p_3 p_1 (a_3 a_1)) \Big]. \quad (46)$$

Equation (46) can be seen as a special case of (43) when $U = 1$, with U as defined in (38). Since all spatial derivatives in (9) and in (43) are confined U, then (45) on diagonal variables, even when not spatially homogeneous, is no more general than a minisuperspace theory.[5] Therefore the restriction of the Ashtekar theory to diagonal variables yields a theory not having spatial derivatives, which is essentially the same as a minisuperspace theory. So the action (9) is equivalent with the diagonally restricted Ashtekar theory only in minisuperspace, for the special case $U = 1$. In the full theory where $U \neq 0$, then this is not so and while (45) is Dirac-inconsistent, equation (9) is a Dirac consistent theory as we have demonstrated. So these two actions are definitely not equivalent on the subspace of diagonal variables in the general case. This then brings in the question of whether there exists action for (9) which for $U \neq 1$ constitutes analogue of the diagonally restricted version of (45), such that the action is not inconsistent in the full theory as is (45). We will relegate the writing down of the desired action to the discussion section of this paper.

5.1 Resolution of the disparity between minisuperspace and the full theory

We will now revisit the question of whether there exists a consistent action analogous to (45), which can be interpreted as the antecedent of the Dirac-consistent action (9). The arguments of the previous section show that in minisuperspace where $U = 1$, equation (9) can be obtained by removing the Gauss' law and diffeomorphism constraints and restricting (32) to diagonal variables. Moreover, (32) leads via these restrictions initially to (45), which is not Dirac consistent in the full theory. Since (9) is a Dirac consistent theory in the full theory, then a pertinent question regards the mechanism by which the Dirac-inconsistent (45) can become associated with the a Dirac-consistent (9) in the general case $U \neq 1$.

[5]This is because there are no spatial derivatives in (46), which moreover is Dirac inconsistent unless the variables are chosen to be spatially homogeneous. The spatial derivatives in (46) have dropped out for the same reason that they drop out of the canonical one form of (39). However recall that (39) still has spatial derivatives contained in $(\det b)^{1/2}$ which multiplies the lapse function N, whereas (45) and (46) do not.

The root cause for the disparity apparently resides in the term U, which contains all spatial derivatives of the theory. Recall that U is contained in (9) but is not contained in (45). There is a certain transformation known as the CDJ Ansatz[6]

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e, \tag{47}$$

where $\Psi_{ae} = \Psi_{(ae)} \in SO(3,C) \times SO(3,C)$ is symmetric, transforms (32) into the action (34) when $(\text{det}B) \neq 0$ and $(\text{det}\Psi) = 0$. Let us examine the implication of (47) for (43) and (39), the 'reduced' versions of (32) and (34) which follow from (9). Note that (39) can be written as

$$I = -\frac{i}{G} \int dt \int_\Sigma d^3x \Big[\Psi_{ae} B^i_e \dot{A}^a_i$$

$$-iN(\text{det}B)^{1/2} \sqrt{\text{det}\Psi} \big(\Lambda + \text{tr}\Psi^{-1}\big)\Big]\Big|_{diag(A);diag(\Psi)}, \tag{48}$$

with phase space restrictions $\Psi_{ae} = \delta_{ae}\Psi_{aa} \equiv \delta_{ae}\lambda_e$ and $A^a_i = \delta^a_i a_a$ to diagonal variables. The unrestricted versrion of (48), namely where the variables can be nondiagonal, is simply the 3+1 decomposition of (34) with the Gauss' law constraint removed. An easy way to see this is to look at the integrand of the canonical one form. First use the following definitions for the components of the curvature

$$B^i_a = \frac{1}{2}\epsilon^{ijk} F^a_{jk}; \quad F^a_{0i} = \dot{A}^a_i - D_i A^a_0, \tag{49}$$

where $D_i v_a = \partial_i v_a + f_{abc} A^b_i v_c$ is the $SO(3,C)$ covariant derivative of the $SO(3,C)$-valued vector v_a. Then defining $\epsilon^{ijk} \equiv \epsilon^{0ijk}$ and using the symmetries of the 4-D epsilon symbol $\epsilon^{\mu\nu\rho\sigma}$, we have

$$\begin{aligned}\Psi_{(ae)} B^i_e \dot{A}^a_i &= \frac{1}{2}\Psi_{(ae)}\epsilon^{ijk} F^e_{jk}(F^a_{0i} + D_i A^a_0) \\ &= \frac{1}{8}\Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + \Psi_{(ae)} B^i_e D_i A^a_0.\end{aligned} \tag{50}$$

The first term on the right hand side of (50) is the same as the first term of (34), which includes the Gauss' constraint. The second term of (50) removes this Gauss' constraint, which can be obtained by integration by parts with discarding of boundary terms $\Psi_{(ae)} B^i_e D_i A^a_0 \rightarrow -A^a_0 B^i_e D_i \Psi_{(ae)}$. The same holds true on the diagonally restricted subspace of this.

Equation (48) is the same as the Dirac consistent theory (9) after the redesignation of variables (35) and (36). But substitution of (47) in conjunction with restriction to diagonal variables transforms (45) into (48). Since

[6]This can be seen as the spatial restriction of one of the equations of motion arising in Plebanski's theory of gravity [8].

(45) under (47) transforms, upon restriction to diagonal variables, into (39), and (39) transforms via canonical transformation into (43), then it follows that (47) is a noncanonical transformation. The conclusion is that this noncanonical transformation, in conjunction with a restriction to diagonal variables (or any of the quantiable configurations Γ_q) is what is necessary to make a Dirac consistent theory out of the reduction (as we have defined it in this paper) of (32). A way to see this is that equation (47) contains spatial derivatives on the right hand side in B^i_a, whereas there are no spatial derivatives explicitly present on the left hand side. It is precisely these derivatives from B^i_a which make the difference between a Dirac-consistent full theory of (9) and a Dirac-inconsistent full-theory of (45).[7]

6 Conclusion and discussion

The main aim of this paper at presenting an action (9) which realizes the Lie subalgebra of temporal coordinate transformations (5) has been carried out.[8] We have presented an action I_{Kin} in equation (9) which has been shown to be Dirac consistent at the classical level and to exhibit two physical degrees of freedom per point. We have shown the relation of I_{Kin} to two formulations of general relativity, namely the Ashtekar variables and a certain antecedent of the CDJ pure spin connection formulation in [7]. In basic terms, the action I_{Kin} can be seen as a restriction of the actions of these formulations to diagonal variables where the Gauss' law and diffeomorphism constraints have been removed by hand. While this is strictly speaking, not technically rigorous as a gauge-fixing procedure, the associated action I_{Kin} is still nevertheless a stand-alone action in the full theory and consistent in the Dirac sense.[9] Hence we would like (9) to serve as a motivation for putting in place a rigorously correct gauge-fixing procedure for full GR. The issue of equivalence of the theories in light of the restrictions, or gauge-choices in certain interpretations, is one which we have reserved for addressal in a subsequent paper.

[7]The latter being Dirac-consistent only in minisuperspace.

[8]This is notwithstanding the fact that there are phase space structure functions appearing in (30) which still need to be interpreted.

[9]For an analogy, the action I_{Ash} for GR in Ashtekar variables [4] can be obtained from Plebanski's I_{Pleb} action [8] in the so-called time gauge, which sets three degrees of freedom coresponding to the choice of a Lorentz frame to zero. But even though $I_{Ash} \subset I_{Pleb}$ is a restriction of Plebanski's action to this specialized sector, the Ashtekar action is still self-consistent in the Dirac sense and is a stand-alone action irrespective of the issue of its equivalence with I_{Pleb}.

7 Appendix A: Quantizable configurations of configuration space

We have shown that the kinematic phase space action (9) can be seen as the diagonal subspace of an action appearing in [7] except with the Gauss' constraint missing. But we have shown that this action is Dirac consistent for a diagonal connection. However, (45) is Dirac consistent only in minisuperspace for a diagonal connection. This leads to the question of whether there are any additional configurations analogous to the diagonal case arising from (9), which are Dirac consistent.

The reason why (9) rather than (39) is in suitable form for canonical analysis is because (39) is not in canonical form. This can be seen from the fact that the variation of its canonical one form, even for the case of a diagonal connection

$$\delta\Big(\int_\Sigma d^3x\lambda_a b^i_a \delta a^a_i\Big)\Big\vert_{diag(A)}$$

$$= \int_\Sigma d^3x\Big[(a_2a_3)\delta\lambda_1\wedge\delta a_1 + \lambda_1\delta(a_2a_3)\wedge\delta a_1\Big] + Cyclic\ Perms, \tag{51}$$

does not yield a closed symplectic 2-form owing to the second term on the right hand side of (51). This difficulty is compounded in the more general case where one is not limited to diagonal variables, which brings spatial derivatives into the symplectic 2-form

$$\delta\boldsymbol{\theta}_{general} = \delta\Big(\int_\Sigma d^3x\lambda_f b^i_f \delta a^f_i\Big) = \int_\Sigma d^3x\Big[b^i_f\delta\lambda_f\wedge\delta a^f_i + \lambda_f(\epsilon^{ijk}D_j\delta a^f_k)\wedge\delta a^f_i\Big]. \tag{52}$$

Equation (52) is not a symplectic two form $\boldsymbol{\Omega}_{general}$ of canonical form $\boldsymbol{\Omega} = \delta(p\delta q) = \delta p\wedge\delta q$, and is not suitable for quantization. The configuration space part of $\boldsymbol{\theta}_{Kin}$ splits into two contributions $b^i_f\delta a^f_i = m_f + n_f$, where

$$m_f = \epsilon^{ijk}(\partial_j a^f_k)\delta a^f_i;\quad n_f = \frac{1}{2}\epsilon^{ijk}f_{fgh}a^g_j a^h_k\delta a^f_i. \tag{53}$$

Note that m_f contains spatial gradients of a^f_i, while n_f is free of spatial gradients. We will see that a sufficient condition for (52) to admit a canonical structure on Ω_{Kin} is that the second term on the right hand side of (52) vanishes, which is tantamount to the requirement that m_f in (53) be zero for all f. Let us determine the configurations a^f_i for which this is the case by expanding m_f and rearranging the terms into the following form

$$m_f = (\partial_2 a^f_3 - \partial_3 a^f_2)\delta a^f_1 + (\partial_3 a^f_1 - \partial_1 a^f_3)\delta a^f_2 + (\partial_1 a^f_2 - \partial_2 a^f_1)\delta a^f_3$$

$$= \big((\delta a^f_2)\partial_3 - (\delta a^f_3)\partial_2\big)a^f_1 + \big((\delta a^f_3)\partial_1 - (\delta a^f_1)\partial_3\big)a^f_2 + \big((\delta a^f_1)\partial_2 - (\delta a^f_2)\partial_1\big)a^f_3. \tag{54}$$

From (54) it is clear that a sufficient condition for $m_f = 0$ is that all except three matrix elements of a^f_i be zero, with the nonzero elements such that no two appear in the same row or column. In other words, we must have $(\det a^f_i) \neq 0$, which restricts the connection to one of the six forms

$$a^a_i = \begin{pmatrix} a^1_1 & 0 & 0 \\ 0 & a^2_2 & 0 \\ 0 & 0 & a^3_3 \end{pmatrix}, \begin{pmatrix} a^1_1 & 0 & 0 \\ 0 & 0 & a^2_3 \\ 0 & a^3_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a^2_1 & 0 \\ a^1_2 & 0 & 0 \\ 0 & 0 & a^3_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a^2_1 & 0 \\ 0 & 0 & a^3_2 \\ a^1_3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a^3_1 \\ a^1_2 & 0 & 0 \\ 0 & a^2_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a^3_1 \\ 0 & a^2_2 & 0 \\ a^1_3 & 0 & 0 \end{pmatrix} \in \Gamma_q,$$

where Γ_q defines what we will refer to as the quantizable configurations of configuration space. Hence for $a^f_i \in \Gamma_q$, we have that $m_f = 0$, and that n_f is given by

$$n_f = \frac{1}{2}\epsilon^{ijk} f_{fgh} a^g_j a^h_k \delta a^f_i = (\det a)(a^{-1})^i_f \delta a^f_i. \qquad (55)$$

It is not difficult to see that each of the six configurations Γ_q leads to a Dirac consistent theory as the diagonal sector we have illustrated in this paper. This constitutes six distinct sectors of the full theory (and not minisuperspace) of reduced general relativity that can be studied.

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ATOMIC GRAVITATIONAL CONSTANT AND THE ORIGIN OF ELEMENTARY MAGNETIC MOMENTS

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Abstract

In the previous published papers it is suggested that square root of ratio of atomic gravitational constant G_A and classical gravitational constant G is equal to the Avogadro number N. If $X_E = 295.0606338$ is the gravitational mass generating number, rest energy of electron is defined as $X_E\sqrt{\frac{e^2}{4\pi\epsilon_0}\frac{c^4}{G_A}}$ and muon and tau masses are fitted accurately. It is also suggested that $\sin\theta_W \cong (\alpha X_E)^{-1}$. In this paper it is suggested that the magnitude of weak force can be considered as $F_w \cong \frac{c^4}{G_A}$. Thus Bohr magneton can be expressed as $\mu_B \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_W}}\sin\theta_W$. Hence $\mu \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_X}}\sin\theta_W$ can be considered as the general expression for magnetic moment where F_X is the characteristic magnitude of the force experienced by the particle. With reference to this expression strongly interacting elementary particles magnetic moment can be given as $\mu \cong \frac{ecR_0}{2}\ \sin\theta_W$ where

R_0 is the characteristic nuclear force radius or size of the strongly interacting particle. With reference to measured proton and neutron magnetic moments, R_0 is close to 1.265 fermi and 0.866 fermi respectively. Fermi's weak coupling constant is coupled with the weak and strong force magnitudes. Finally it is noticed that ratio of electroweak energy scale and electron rest energy is equal to the strong and weak force ratio.

Keywords: Classical gravitational constant, atomic gravitational constant, Avogadro number, grand unification, electron rest mass, weak coupling angle, weak force magnitude, electron magnetic moment, strong force magnitude, proton magnetic moment, neutron magnetic moment, Fermi's weak coupling constant and electroweak energy scale.

1 Introduction

Stephen Hawking [1] - in his famous book- says: It would be very difficult to construct a complete unified theory of everything in the universe all at one go. So instead we have made progress by finding partial theories that describe a limited range of happenings and by neglecting other effects or approximating them by certain numbers. (Chemistry, for example, allows us to calculate the interactions of atoms, without knowing the internal structure of an atomic nucleus.) Ultimately, however, one would hope to find a complete, consistent, unified theory that would include all these partial theories as approximations, and that did not need to be adjusted to fit the facts by picking the values of certain arbitrary numbers in the theory. The quest for such a theory is known as 'the unification of physics'. Einstein spent most of his later years unsuccessfully searching for a unified theory, but the time was not ripe: there were partial theories for gravity and the electromagnetic force, but very little was known about the nuclear forces. Moreover, Einstein refused to believe in the reality of quantum mechanics, despite the important role he had played in its development.

Abdus Salam [2], David Gross [3] and Tilman Sauer [4] presented their views on Einstein's works in unification. As the culmination of his life work, Einstein wished to see a unification of gravity and electromagnetism as aspects of one single force. In modern language he wished to unite electric charge with the gravitational charge (mass) into one single entity. Further, having shown that mass the 'gravitational charge' was connected with space-time curvature, he hoped that the electric charge would likewise be so connected with some other geometrical property of space-time structure. Einsteins goal was to generalize general relativity to include electromagnetism. If one wishes to unify electroweak, strong and gravitational interactions it is a must to implement the classical gravitational constant G in the sub atomic physics. By any reason if one implements the planck scale in elementary particle physics and nuclear physics automatically G comes into subatomic physics.

Einstein, more than any other physicist, untroubled by either quantum uncertainty or classical complexity, believed in the possibility of a complete, perhaps final, theory of everything. He also believed that the fundamental laws and principles that would embody such a theory would be simple, powerful and beautiful. The 'old one', that Einstein often referred to, has exquisite taste. After his enormous success at reconciling gravity with relativity, Einstein was troubled by the remaining arbitrariness of the theoretical scheme. First, the separate existence of gravitation and electromagnetism was unacceptable. According to his philosophy, electromagnetism must be unified with general relativity, so that one could not simply imagine that it did not exist. Furthermore, the existence of matter, the mass and the charge of the electron and the proton (the only elementary particles recognized back in the 1920s), were arbitrary features. One of the main goals of a unified theory should be to explain the existence and calculate the properties of matter.

After sometime in the late 1920s Einstein became more and more isolated from the mainstream of fundamental physics. To a large extent this was due to his attitude towards quantum mechanics, the field to which he had made so many revolutionary contributions. Einstein, who understood better than most the implications of the emerging interpretations of quantum mechanics, could never accept it as a final theory of physics. He had no doubt that it worked, that it was a successful interim theory of physics, but he was convinced that it would be eventually replaced by a deeper, deterministic theory. We now have direct evidence for the unification of all forces dreamed by Einstein. Perhaps the most important feature of the extrapolation of the standard models forces is that the energy at which they appear to unify is very close, if not identical, to the point at which gravity becomes equally strong. This indicates that the next stage of unification should include, as Einstein expected, unification of the non-gravitational forces and gravity.

David Gross says: Because of his opposition to quantum mechanics he allowed himself to ignore most of the important developments in fundamental physics for over twenty five years, as Einstein himself admitted in 1954, 'I must seem like an ostrich who buries its head in the relativistic sand in order not to face the evil quanta'. If there is one thing that I fault Einstein for, it is his lack of interest in the development of quantum field theory. To be sure many of the inventors of quantum field theory were soon to abandon it when faced with ultraviolet divergences, but it is hard to understand how Einstein, could not have been impressed with the successes of the marriage of his children quantum mechanics and special relativity. The Dirac [5] equation and quantum electrodynamics had remarkable successes, especially the prediction of anti-particles. How could Einstein not have been impressed?

Kaluza and Klein in 1922 to 1926 showed that if one assumed general relativity in five dimensions, where one dimension was curled up, the resulting theory would look like a four-dimensional theory of electromagnetism and gravity. Electromagnetism emerged as a consequence of gravity in five dimensions. In string theory there are six or seven extra-spatial dimensions. One can imagine that these are curled up to form a small manifold, and remarkably such six dimension compactifications (achieved by solving the generalization of Einsteins equations in ten dimensions) can produce a world remarkably like our own, in which the shape of the extra dimensions determines the complete matter content and all the forces of nature, as seen by a four-dimensional observer.

To unify 2 interactions if 5 dimensions are required, for unifying 4 interactions 10 dimensions are required. For 3+1 dimensions if there exists 4 (observed) interactions, for 10 dimensions there may exist 10 (observable) interactions. To unify 10 interactions 20 dimensions are required. This

logic seems to indicate that with 'n' new dimensions one may not be able to resolve the problem of unification. More over new problems and new properties will come into picture and makes the 4 dimensional unification program more complicated. Right now quantitatively and qualitatively:
1) one cannot implement the planck scale in 'atomic' and 'nuclear' space.
2) one cannot think about the 'reduced magnitudes' of quantized elementary charge or angular momentum.

2 Coulomb mass and its magnetic moment

The first step in unification is to understand the origin of the rest mass of a charged elementary particle. Second step is to understand the combined effects of its electromagnetic (or charged) and gravitational interactions. Third step is to understand its behaviour with surroundings when it is created. Fourth step is to understand its behaviour with cosmic space-time or other particles. Right from its birth to death, in all these steps the underlying fact is that whether it is a strongly interacting particle or weakly interacting particle, it is having some rest mass. To understand the first 2 steps somehow one must implement the gravitational constant in sub atomic physics.

The subject of unification is not new. But here the fundamental question to be answered is: without a 'mass content' can electric charge preserve its individual identity? For example even though 'rest mass' of photon is zero it possesses 'energy'. For any elementary charged massive particle - which is more fundamental either the 'mass' or the 'charge? Here authors humble opinion is: charge can be considered as the fundamental, inherent and characteristic property of the charged massive particle. For the same magnitude of charge, proton's mass is 1836.15 times heaver than the mass of electron. Observed elementary mass spectrum ranges from 0.511 MeV to 182 GeV. But very interesting and surprising observation is that magnitude of charge remains at e or $2e$. How to understand this situation? Concept of quantization of charge states that- in nature 'charge' exists only in integral multiples of e.

If 'charge' is the inherent property and 'mass' is the induced or secondary property then the fundamental question to be answered is: how to understand the origin of particles magnetic moment? Till now quantitatively or qualitatively either the large number hypothesis or the string theory or the planck scale is not implemented in particle physics. Unifying gravity with the other three interactions would form a theory of everything (TOE), rather than a GUT. In the published papers and book Seshavatharam and Lakshminarayana [6-12] proposed that there may exist coulomb's charged particle of mass-energy

$$M_c c^2 \cong \sqrt{\frac{e^2}{4\pi\epsilon_0}\left(\frac{c^4}{G}\right)} \cong 1.042940852 \times 10^{18}\ GeV. \tag{1}$$

$$M_c \cong \sqrt{\frac{e^2}{4\pi\epsilon_0 G}} \cong 1.859210775 \times 10^{-9}\ Kg. \tag{2}$$

If $m_p = 1.672621638 \times 10^{-27}\ Kg$ = rest mass of proton, $m_e = 9.109382154 \times 10^{-31}\ Kg$ = rest mass of electron, N= Avogadro number and G=Gravitational constant, semi empirically it is noticed that

$$\ln\left(\frac{M_c}{m_p}\right) \cong \sqrt{\frac{m_p}{m_e} - \ln(N^2)}. \tag{3}$$

Here, Lhs = 41.55229152; Rhs = 41.55289244; A very beautiful fit. In grand unification program this type of fitting should not be ignored. Considering all the atomic physical constants, obtained value of the gravitational constant is $6.666270179 \times 10^{-11}\ m^3 Kg^{-1} sec^{-2}$. This is a very nice fitting. How to interpret this strange relation? Please note that absolute lab measurements of G have been made only on scales about 1 cm to 1 meter only. Amedeo Carlo Avogadro [13] proposed his hypothesis in 1811. P.J. Mohar and B.N. Taylor [14] recommended a value of $N \cong 6.022141793 \times 10^{23}$ and $G \cong 6.6742867 \times 10^{-11}\ m^3 Kg^{-1} sec^{-2}$.

Till today there is no explanation for the origin of large Avogadro number. The Avogadro constant expresses the number of elementary entities per mole of substance. Avogadro's constant is a scaling factor between macroscopic and microscopic (atomic scale) observations of nature. It is an observed fact. The very unfortunate thing is that even though it is a large number it is neither implemented in cosmology nor implemented in grand unification. Note that ratio of planck mass and electron mass is $\frac{N}{8\pi}$.

The beauty of above expression (2) is that it generates a 'mass content' from e and G. In the sense it is generating 'inertia' in the free space. Here the fundamental questions to be answered are: from where elementary charge is coming into picture? How and why it exists in the universe? How many elementary charges are there in the universe? Is 'coulomb mass' the mother of all the observed charged and neutral elementary massive particles? Qualitatively this obtained mass unit play some role in the generation of elementary particle's rest mass. But from numerical point of view this mass is very large compared to the observed elementary particle's rest mass.

The sources of magnetic fields, down to the atomic scale, are electrical currents. A small current loop appears from a distance as a magnetic dipole. Thus orbiting electrons give to an atom a magnetic dipole moment associated with its orbital angular momentum. In addition, subatomic particles also have magnetic dipole moments. The magnetic dipole moment of a nucleus comes in part from the proton currents within it, and the magnetic dipole moment of a proton (and other baryons, and mesons) comes in part from quark currents within it. In addition, the electron and the up and down quarks also have an 'intrinsic' magnetic moment associated with their spin, although they are supposed to be point-like; the heavier leptons and quarks have one too. In a naive classical picture, the electron continually goes in circles around itself. The Dirac equation endows the electron with a spin $\frac{1}{2}\hbar$ and a magnetic moment $\frac{e\hbar}{2m_e}$. Both arise from quantum relativistic effects that are built into Dirac's theory. Beyond that, quantum electrodynamics predicts that the intrinsic magnetic moment of the electron is actually $\mu \cong 1.00116\mu_B$ in stunning agreement with experiment. Generally, the magnetic dipole moments of elementary particles are understood much better than their masses. With reference to Dirac's 'magnetic moment' concepts above expressed mass unit's magnetic moment can be expressed as

$$\mu \cong \frac{e\hbar}{2M_c} \cong \frac{e\hbar}{2}\sqrt{\frac{4\pi\epsilon_0 G}{e^2}} \cong \frac{\hbar}{2}\sqrt{4\pi\epsilon_0 G} \cong 4.54389 \times 10^{-45}\ J.T^{-1}. \quad (4)$$

Qualitatively this idea couples gravity, electromagnetism and quantum mechanics. How to understand this? But quantitatively its magnitude is 10^{21} times smaller than the Bohr magneton. These expressions for 'grand unified mass' and 'grand unified magnetic moment' indicates that grand unification is not far from reality provided there exists a large scaling factor. It can be supposed that elementary particles construction is much more fundamental than the black hole's construction. Till today in the laboratory no such a particle is observed with such a large mass or no such a small magnetic moment is also observed. To move from this large mass unit to the electron mass one must consider some type of large coupling constant or a proportionality number or a scaling factor. Now the real problem comes into picture.

To have a small mass unit one cannot assume that small massive particle possesses a fractional magnitude of e. In CGS system of units value of $4\pi\epsilon_0$ is unity. The only one alternative that can be allowed is variation of G. Please note that the only one gravitational physical constant is: Newton's gravitational constant. Note that in the atomic or nuclear physics, till today no one measured the gravitational force of attraction between the proton and electron and experimentally no one measured the value of the gravitational constant. Physicists say - if strength of strong interaction is unity, with reference to the strong interaction, strength of gravitation is 10^{-39}. Whether the nature of variation is cosmic or there exists two kinds of gravitational constants one for the classical physics and the other for the atomic system- has to be analysed.

From above observations it can be suggested that for unifying gravity, electromagnetism and quantum mechanics a 'large value of fixed gravitational constant' is required. Some attempts have been done in physics history. The large value of gravitational constant can be called as 'atomic gravitational constant' or 'strong gravitational constant'. The idea of strong gravity originally referred specifically to mathematical approach of Abdus Salam [15-17] of unification of gravity and quantum chromo-dynamics, but is now often used for any particle level gravity approach. For defining the atomic gravitational constant a large scaling factor is required. If the scaling factor is a known one, then to some extent - its historical data and physics background makes and brings the unification concepts into one stream. Compared to the current research - it may be in the main streamline or secondary streamline - it can be decided by the future thoughts and experiments.

3 Grand unification and the need of atomic gravitational constant

The strong or atomic gravitational constant is the supposed physical constant of strong gravitation, involved in the calculation of the gravitational attraction at the level of elementary particles and atoms. From the standpoint of 'infinite hierarchical nesting of matter' and Le Sage's theory of gravitation, the presence of two gravitational constants shows the difference between the properties of gravitons and properties of matter at different levels of matter. The strong gravitational constant is also included in the formula describing the nuclear force through strong gravitation and torsion field of rotating particles. A feature of the gravitational induction is that if two bodies rotate along one axis and come close by the force of gravitation, then these bodies will increase the angular velocity of its rotation. In this regard, it is assumed that the nucleons in atomic nuclei rotate at maximum speed. This may explain the equilibrium of the nucleons in atomic nuclei as a balance between the attractive force of strong gravitation and the strong force of the torsion field (of gravito-magnetic forces in gravito-magnetism).

Qualitatively it is also thought that the magnetic moment of the proton is created by the maximum rotation of its positive charge distributed over the volume of the proton in the form of a ball, when the centripetal acceleration at the equator becomes equal to acceleration of strong gravitation. In literature one can refer the beautiful works of Abdus Salam, C. Sivaram, Sabbata, A.H. Chamseddine, J. Strathdee, Usha Raut, K. P. Sinha, Perng. J.J, Recami, Robert L. Oldershaw, K.Tennakone, S.I Fisenko and S.G.Fedosin ([18]-[30]). Various proposed values of the strong gravitational constant are 2.06×10^{25}, 6.7×10^{27}, 2.18×10^{28}, 2.4×10^{28}, 3.9×10^{28}, 1.514×10^{29}, 3.2×10^{30}, 5.1×10^{31}, 6.9×10^{31} and 2.77×10^{32} $m^3 Kg^{-1} sec^{-2}$. In this connection authors in the previous papers suggested that square root of ratio of atomic gravitational constant and classical gravitational is equal to the Avogadro number N. Value of the proposed atomic gravitational constant is equal to $G_A \cong N^2G \cong 2.420509614 \times 10^{37}\ m^3Kg^{-1}sec^{-2}$. For each and every elementary particle its corresponding value of G_m can be expressed as

$$G_m \cong \frac{e^2}{4\pi\epsilon_0 m_x^2}. \tag{5}$$

Here G_m = magnitude of G corresponding to the mass of the particle m_x. The interesting point to be noted is that unlike the classical or continuous mass range of celestial massive bodies, elementary particles mass spectrum follows certain quantum rules and hence there exists some governing procedure for the observed mass spectrum. Not only that each interaction is having some coupling constants. Considering leptons three exists only one basic particle- that is electron. Considering hadrons there exists only one stable particle - that is proton. Hence value of G_m can be fixed. If one is able to inter change the coupling constants , there is a possibility of fixing the value of G_m. In this way this proposed idea differs from Dirac's proposal of variation of G with cosmic time. Based on Sciama's proposal ([31],[32]) in atomic and nuclear physics, with reference to the nuclear mass and size, magnitude of the nuclear characteristic gravitational constant can be given as

$$G_m \cong \frac{R_p c^2}{m_p}. \tag{6}$$

Here, m_p = mass of proton, R_p = size of proton.

To bring down the planck mass scale to the observed elementary particles mass scale a large scale factor is required. Just like relative permeability and relative permittivity by any suitable reason in atomic space if one is able to increase the value of classical gravitational constant, it helps in four ways. Observed elementary particles mass can be generated and grand unification can be achieved. Electromagnetism, quantum mechanics and gravity can be studied in a unified manner. Third important application is characteristic building block of the cosmological 'dark matter' can be quantified in terms of fundamental physical constants. Joshua A. Frieman et al [33] discussed about the origin of dark matter. Fourth important application is - no extra dimensions are required. Finally nuclear physics and quantum mechanics can be studied in the view of 'strong nuclear gravity' where nuclear charge and atomic gravitational constant play a crucial role in the nuclear space-time curvature, QCD and quark confinement. Not only that cosmology and particle physics can be studied in a unified way.

Whether it may be real or an equivalent if it is existing as a 'single constant' its physical significance can be understood. Charged lepton masses can be fitted. Hence their magnetic moments can be understood. 'Nuclear size' can be fitted with 'nuclear Schwarzschild radius'. 'Nucleus' can be considered as 'strong nuclear black hole'. Nuclear binding energy constants can be generated directly. Proton-neutron stability can be studied. Origin of 'strong coupling constant' and 'Fermi's weak coupling constant' can be understood. Chris Quigg [34] and J. Erler et al [35] discussed about the estimation of strong coupling constant and Fermi's weak coupling constant. Authors feel that these applications can be considered favourable for the proposed assumptions and further analysis can be carried out positively for understanding and developing this proposed 'Avogadro's strong nuclear gravity'.

4 Planck mass and the electron mass

It is noticed that ratio of planck mass and electron mass is 2.389×10^{22} and is 25.2 times smaller than the Avogadro number. It is also noticed that the number 25.2 is close to $8\pi \cong 25.13274$. Qualitatively this idea implements gravitational constant in particle physics. C. Brans and R.H. Dicke discussed [36] about the planck mass in detail. Note that planck mass is the heaviest mass and neutrino mass is the lightest mass in the known elementary particle mass spectrum.

$$\frac{M_P}{m_e} \cong \sqrt{\frac{\hbar c}{G m_e^2}} \cong 2.3892245954 \times 10^{22} \cong \frac{N}{8\pi}. \tag{7}$$

Here, M_P = planck mass and m_e = electron rest mass. Hence electron rest mass can be expressed as

$$m_e \cong \frac{8\pi}{N}\sqrt{\frac{\hbar c}{G}} \cong 8\pi\sqrt{\frac{\hbar c}{N^2 G}} \cong 9.083115709 \times 10^{-31}\ Kg. \tag{8}$$

Accepted value of $m_e = 9.109382154 \times 10^{-31}\ kg$ and accuracy is 99.7116%.

In terms of the above introduced 'coulomb' mass unit it can be expressed as

$$m_e \cong \frac{8\pi}{N\sqrt{\alpha}}\sqrt{\frac{e^2}{4\pi\epsilon_0 G}} \cong \frac{8\pi}{\sqrt{\alpha}}\sqrt{\frac{e^2}{4\pi\epsilon_0\left(N^2G\right)}}. \tag{9}$$

Here it can be assumed that- if $\frac{8\pi}{\sqrt{\alpha}} \cong 294.2098$ is the electromagnetic mass induction or generation strength then $N^2G \cong G_A$ can be considered as the atomic gravitational constant. In grand unification program this number

$$X_E \cong \frac{8\pi}{\sqrt{\alpha}} \cong \sqrt{\frac{4\pi\epsilon_0\left(N^2G\right)m_e^2}{e^2}} \cong 295.0606338. \tag{10}$$

can be called as the lepton-quark-nucleon gravitational mass generator. It is the utmost fundamental ratio compared to the fine structure ratio α. It plays a vital role in particle physics. Here the important and interesting observation is that

$$m_X \cong \sqrt{\frac{e^2}{4\pi\epsilon_0\left(N^2G\right)}} \cong 3.087291597 \times 10^{-33}\ Kg. \tag{11}$$

$$m_X c^2 \cong \sqrt{\frac{e^2c^4}{4\pi\epsilon_0\left(N^2G\right)}} \cong \sqrt{\frac{e^2}{4\pi\epsilon_0}\left(\frac{c^4}{N^2G}\right)} \cong 1.731843735\ KeV. \tag{12}$$

This mass unit is very close the (neutral) neutrino mass. Conceptually this can be compared with the charged dark matter. The fundamental question to be answered is : 1.7318 keV is a potential or a charged massive particle? If it is a particle its pair annihilation leads to radiation energy. If it is the base particle in elementary particle physics - observed particle rest masses can be fitted. Authors humble opinion is: it can be considered as the basic charged lepton or lepton potential. It can be considered as the basic charged dark matter candidate. Using this mass unit and above defined number X_E, muon and tau masses can be fitted as

$$m_l c^2 \cong \left[X_E^3 + \left(n^2X_E\right)^n\sqrt{N}\right]^{\frac{1}{3}}\sqrt{\frac{e^2c^4}{4\pi\epsilon_0 G_A}} \cong \frac{2}{3}\left[E_c^3 + \left(n^2X_E\right)^n E_a^3\right]^{\frac{1}{3}}. \tag{13}$$

Here n =0, 1 and 2. E_c = coulombic energy constant and E_a = asymmetry energy constant of the semi empirical mass formula respectively. P. Roy Chowdhury et al [37] modified the semi-empirical mass formula. At n = 0, electron mass is defined, At n=1, obtained muon mass is 105.95 MeV and at n =2 obtained tau mass is 1777.4 MeV. At n =3 predicted mass is 42262 MeV. In terms of the atomic gravitational constant = G_A, atomic planck mass can be represented as

$$m_P \cong \sqrt{\frac{\hbar c}{(N^2 G)}} \cong 3.614056909 \times 10^{-32}\ Kg. \tag{14}$$

$$m_P c^2 \cong \sqrt{\frac{\hbar c^5}{(N^2 G)}} \cong \sqrt{\hbar c \left(\frac{c^4}{N^2 G}\right)} \cong 20.27337431\ KeV. \tag{15}$$

5 Atomic or nuclear weak force magnitude

In classical physics or in cosmology or in black hole physics or in planck scale physics, the operating force limit is $\left(\frac{c^4}{G}\right)$. Seshavatharam [38] discussed about its role in Black hole physics and W. C. Daywitt [39] discussed about its role in Planck vacuum. Similar to this, the characteristic force limit in atomic or nuclear physics can be given as $\left(\frac{c^4}{G_A}\right)$. It can be expressed as

$$\frac{c^4}{G_A} \cong \frac{c^4}{N^2 G} \cong 3.337152088 \times 10^{-4}\ newton. \tag{16}$$

It can be suggested that $\frac{c^4}{G_A} \cong F_w$ can be called as the weak force magnitude. This weak force is responsible for the nuclear weak decay. In nuclear physics, if F_s represents the magnitude of strong force it is noticed that

$$\sqrt{\frac{F_s}{F_w}} \cong 2\pi \ln\left(\frac{G_A}{G}\right) \cong 2\pi \ln\left(N^2\right). \tag{17}$$

where $F_s \cong 157.9944058$ *newton* can be called as the magnitude of the nuclear strong force. Giger H and Marsden E [40] and Robert Hofstadter [41] explained the methods of estimating the nuclear size. Hence characteristic nuclear size R_0 can be expressed as

$$R_0 \cong \sqrt{\frac{e^2}{4\pi\epsilon_0 F_s}} \cong 1.208398568 \times 10^{-15}\ m. \tag{18}$$

Claudia Glassman [42] discussed about the measurement of strong coupling constant at HERA. If α_s is the strong coupling constant, it is also noticed that,

$$F_s \cong \frac{e^2}{4\pi\epsilon_0 R_0^2} \cong e^{\frac{1}{\alpha_s}} \times \frac{2E_a}{E_c} \times F_w. \tag{19}$$

$$\frac{1}{\alpha_s} \cong \ln\left(\frac{E_c}{2E_a} \times \frac{F_s}{F_w}\right). \tag{20}$$

Note that $\frac{2E_a}{E_c}$ plays a crucial role in nuclear stability as

$$Z_S \cong \frac{A}{2 + \left(\frac{E_c}{2E_a}\right) A^{\frac{2}{3}}}. \tag{21}$$

where A is the mass number and Z_s is the proton number. Please note that for getting stability, neutron of the unstable isotope emits beta particle and becomes a stable one. In hydrogen atom, force of attraction between proton and electron can be represented as,

$$\frac{e^2}{4\pi\epsilon_0 a_0^2} \cong \left(\frac{E_c}{2E_a}\right)^2 F_w \tag{22}$$

Here a_0 is the Bohr radius. It can be expressed as

$$a_0 \cong \frac{2E_a}{E_c}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_w}}. \tag{23}$$

Hence potential energy of electron in hydrogen atom can be given as

$$\frac{e^2}{4\pi\epsilon_0 a_0} \cong \left(\frac{E_c}{2E_a}\right)\sqrt{\frac{e^2 F_w}{4\pi\epsilon_0}} \cong \alpha^2 m_e c^2. \tag{24}$$

Here α is the fine structure ratio. G. P. Shpenkov [43] explained the meaning of fine structure ratio. Giving importance to the phenomena of β -decay, rest mass-energy of electron can be expressed as

$$m_e c^2 \cong \frac{1}{\alpha^2} \times \frac{E_c}{2E_a} \times \sqrt{\frac{e^2 F_w}{4\pi\epsilon_0}} \cong X_E \times \sqrt{\frac{e^2 F_w}{4\pi\epsilon_0}}. \tag{25}$$

Hence in Hydrogen atom force on electron can be expressed as

$$\frac{e^2}{4\pi\epsilon_0 a_0^2} \cong \left(X_E \alpha^2\right)^2 \frac{c^4}{G_A} \cong \left(X_E \alpha^2\right)^2 F_w. \tag{26}$$

If F_e is the electromagnetic force on electron in hydrogen atom, similar to the square root of ratio of strong force and weak forces, square root of ratio of electromagnetic force and weak forces can be represented as

$$\sqrt{\frac{F_e}{F_w}} \cong \frac{E_c}{2E_a} \cong X_E \alpha^2. \tag{27}$$

6 Weak force and the magnetic moment of electron

P. A.M. Dirac [5], Richard Fyenman, R.L.Mills, G. Gabrielse, Boyer, W.K.H. Panofsky, J.D. Jackson, D.J. Giffiths, A.O. Barut, M.Rivas, P.Kusch and G.P. Shpenkov presented a clear picture of elementary particles geometry, mass and magnetic moments ([44]-[55]). From above expressions magnetic moment of electron can be expressed as

$$\mu_e \cong \frac{e\hbar}{2m_e} \cong \frac{e}{2\alpha X_E} \sqrt{\frac{e^2 G_A}{4\pi\epsilon_0 c^2}} \cong \frac{ec}{2\alpha X_E} \sqrt{\frac{e^2}{4\pi\epsilon_0 F_w}}. \tag{28}$$

In our previous papers [6-12] it is suggested that $\alpha X_E \cong 2.153161465$ can be considered as inverse of the weak coupling angle $\sin\theta_W$. Then above expression can be written as

$$\mu_e \cong \frac{e^2}{2} \sqrt{\frac{\mu_0 G_A}{4\pi}} \sin\theta_W \cong \frac{ec}{2} \sqrt{\frac{e^2}{4\pi\epsilon_0 F_w}} \sin\theta_W. \tag{29}$$

From this it is very clear that weak force is responsible for the origin of magnetic moment of electron. For muon and tau their magnetic moments can be expressed as

$$\mu \cong \left(\frac{m_e}{m}\right)\frac{e^2}{2}\sqrt{\frac{\mu_0 G_A}{4\pi}}\sin\theta_W \cong \left(\frac{m_e}{m}\right)\frac{ec}{2}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_w}}\sin\theta_W. \qquad (30)$$

where m is the mass of muon or tau.

7 Strong force and the magnetic moments of proton and neutron

Using the above expression general expression for magnetic moments can be expressed as

$$\mu \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_X}}\sin\theta_W. \qquad (31)$$

where F_X can be referred to the particles characteristic force. A.W. Thomas, G. Sardin, H.J. Lipkin, Y.K. Gambhir, N. Kaiser, Xiang-Song Chen, V. Dimitrsinovic, G. P. Shpenkov, S. J. Dong, N. Mathur, G. L. Strobel, B. Lee Roberts, MA Wei-Xing, W. R. B. de Ara ujo discussed about the baryon magnetic moments ([56]-[70]). Hence for strongly interacting particles magnetic moment can be expressed as

$$\mu \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_s}}\sin\theta_W. \qquad (32)$$

From nuclear electron scattering experiments,

$$\sqrt{\frac{e^2}{4\pi\epsilon_0 F_s}} \cong R_0 \cong 1.21 \text{ to } 1.25\ fermi. \qquad (33)$$

Experiments suggests that proton radius is close to R_p= 0.86 fermi. Sangita Haque et al [71] and B. Ketzer [72] discussed about the radius of proton. Considering these radii- in strong interaction, (32) can be written as

$$\mu \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\epsilon_0 F_s}}\sin\theta_W \cong \frac{ecR_0}{2}\sin\theta_W. \qquad (34)$$

At R_0 =1.21 fermi , $\mu \cong 1.3496 \times 10^{-26}$ J/T and can be compared with the magnetic moment of proton $= 1.41 \times 10^{-26}$ J/T. At $R_0 = 0.86$ fermi, $\mu \cong 9.592 \times 10^{-27}$ J/T and can be compared (neglecting the -ve sign) with the magnetic moment of neutron 9.66×10^{-27} J/T. With reference to the measured proton and neutron magnetic moments, $R_0 = 1.265$ fermi and 0.866 fermi respectively. Compared to the existing methods of estimation of nucleons magnetic moments this method is simple and accurate. Not only that this can be easily applied to other baryons or resonances. Just by guessing the magnitude of the strong force or by guessing the baryon size its magnetic moment can be estimated.

8 Fermi's weak coupling constant G_F and the electroweak energy scale

It is noticed that, Fermi's weak coupling constant G_F depends on the magnitudes of weak force and strong force. Semi empirically it is noticed that

$$G_F \propto \frac{F_w}{F_s} \times \frac{\hbar c}{2} \times \frac{e^2}{4\pi\epsilon_0 F_s}. \tag{35}$$

It is noticed that,

$$\frac{F_w}{F_s} \times \frac{\hbar c}{2} \times \frac{e^2}{4\pi\epsilon_0 F_s} \cong 4.87552608 \times 10^{-62}\ J.m^3. \tag{36}$$

Recommended value ([14],[34]) of $G_F \cong 1.435841042 \times 10^{-62}$ $J.m^3$ and $\frac{G_F}{\hbar^3 c^3} \cong 1.166371 \times 10^{-5}\ GeV^{-2}$. Qualitatively and quantitatively with reference to $\sin\theta_W \cong \frac{1}{\alpha X_E} \cong 0.464433353$, with an error of 3.58%, G_F can be expressed as

$$G_F \cong \sqrt{2}\sin^2\theta_W \times \frac{F_w}{F_s} \times \frac{\hbar c R_0^2}{2} \cong \frac{\sin^2\theta_W}{\sqrt{2}} \times \frac{F_w}{F_s} \times \hbar c R_0^2. \tag{37}$$

where $F_w \cong \frac{c^4}{N^2 G}$, $\sqrt{\frac{F_s}{F_w}} \cong 2\pi \ln\left(\frac{G_A}{G}\right)$, $\sqrt{\frac{e^2}{4\pi\epsilon_0 F_s}} \cong R_0 \cong 1.208399\ fermi$. If so, charged weak boson rest energy can be expressed as

$$Wc^2 \cong \frac{1}{\sqrt{2}} \times \sqrt{\frac{F_s}{F_w}} \times \frac{\hbar c}{R_0} \cong 79.45\ GeV. \tag{38}$$

Neutral weak boson rest energy can be expressed as

$$Zc^2 \cong \frac{Wc^2}{\cos\theta_W} \cong \frac{1}{\sqrt{2}\cos\theta_W} \times \sqrt{\frac{F_s}{F_w}} \times \frac{\hbar c}{R_0} \cong 89.7124\ GeV. \tag{39}$$

Presently believed electroweak energy scale can be expressed as

$$E_W \cong \sqrt{\frac{\hbar^3 c^3}{\sqrt{2}G_F}} \cong \frac{1}{\alpha \sin\theta_w} \times \sqrt{\frac{F_s}{F_w}} \times \sqrt{\frac{e^2 F_s}{4\pi\epsilon_0}} \cong 241.9277486\ GeV. \tag{40}$$

Eliminating $\sin\theta_w$ it can also be expressed as

$$E_W \cong \sqrt{\frac{\hbar^3 c^3}{\sqrt{2}G_F}} \cong X_E \times \frac{F_s}{F_w} \times \sqrt{\frac{e^2 F_w}{4\pi\epsilon_0}} \cong \frac{F_s}{F_w} \times m_e c^2. \tag{41}$$

where $m_e c^2$= 0.511 MeV = rest energy of electron. This is a very simple and strange equation and couples the four fundamental nuclear interactions. This equation is true when $F_w \cong \frac{c^4}{N^2 G}$ and $\sqrt{\frac{F_s}{F_w}} \cong 2\pi \ln\left(\frac{G_A}{G}\right)$. This coincidence clearly establishes the trueness of the magnitude of the proposed weak and strong forces and reality of the existence of the atomic gravitational constant $G_A \cong N^2 G$ in atomic and nuclear physics.

Conclusion

Right from Dirac's theory to the present QCD methods of estimation of elementary particles magnetic moments, proposed method is simple, accurate and throws light on grand unification. It is very clear that leptons magnetic moments depends on the weak force and baryons magnetic moments depends on the strong force. With reference to the proposed weak force and its applications: existence of the atomic gravitational constant can be confirmed. In this new direction authors are working in understanding the neutron and electron mass ratio. Authors humbly request the world science community to kindly look into this new approach.

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ADDITIONAL CONFIRMATION OF "INTERMEDIATE CONTROLLED NUCLEAR FUSION" WITHOUT HARMFUL RADIATIONS OR WASTE

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Abstract

In this paper, we report three tests providing additional experimental confirmations of the recently achieved and verified *Intermediate Controlled Nuclear Fusions* (ICNF). Thanks to various chemical analyses performed by independent laboratories, the first test established the ICNF of silica from carbon and oxygen; the second test confirmed the preceding results; and the third test established the ICNF of oxygen from helium and carbon.

PACS 25.70.Jj, 24.10.-i, 25.70.-z

1. Introduction

Following decades of studies for the prior development of mathematical, physical and chemical formulations as structurally irreversible over time as the energy releasing processes that have to be described (see review [1] and general presentations [2]), and as a result of extensive tests and experimentations conducted for years, in the preceding paper [3] we released, apparently for the first time, experimental evidence on the "existence" of *Intermediate Controlled Nuclear Fusions* (ICNF) whose primary features are the following:

1) *Lack of emission of harmful radiations (such as n, p, α, etc.) and lack of release of radioactive waste.* This fundamental feature is achieved by conceptually and technically restricting the syntheses to light, natural and stable elements.

2) *Control of the fusions via multiple means.* This second important feature is achieved via the control of power, temperature, pressure, flow and other engineering means.

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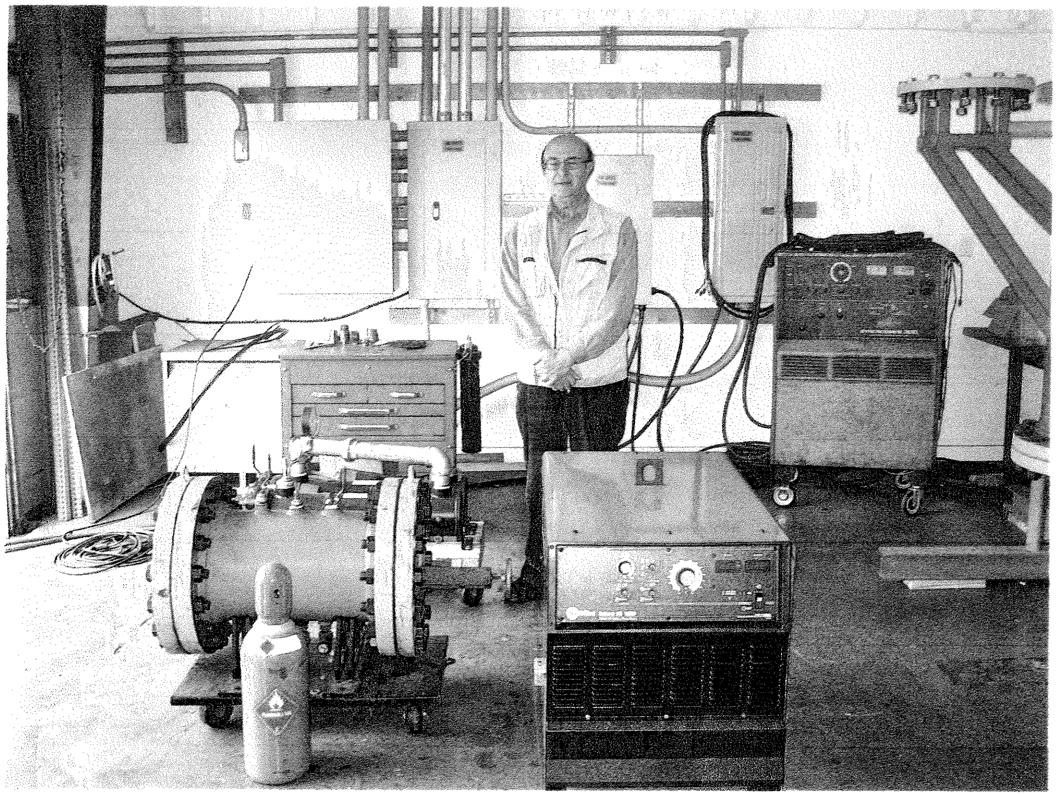

Figure 1: A view of the author with the equipment used for the synthesis of nitrogen from carbon and deuterium [3] showing from the r.h.s.:. the Miller Dimension 1000 AC-DC converter; the pressure bottle of 99.99 pure deuterium; and the carbon steel, 12" x 24" schedule 80 hadronic reactor.

3) *Intermediate character between the so-called hot and cold fuions,* in the sense that the used temperature has values in between the high temperatures of the hot fusion and the low temperature of the cold fusion.

ICNF are achieved via the use of specially constructed, high pressure, steel vessels known as *hadronic reactors* because conceived and constructed via the laws of hadronic mechanics and chemistry [1,2]. Their main function is that of delivering a DC electric arc between suitably selected electrodes submerged within a suitably selected gas at pressure. Under the condition that, for selected electrodes, the gas does allow ICNF, it is called *hadronic fuel.* All tests herein considered deal with hadronic fuels suitably selected to achieve ICNF when traversed by a DC arc between carbon electrodes.

In particular, paper [3] presented the following ICNF

$$D(2,1,1^+,2.0141) + C(12,6,0^+,12.0000) + TR \rightarrow$$
$$\rightarrow N(14,7,1^+,14.0030) + \Delta E, \quad (1a)$$
$$\Delta E = (E_{car} + E_{deu}) - E_{nitr} = 0.0111 \ u, \quad (1b)$$

where TR stands for the *trigger,* namely, an external action (such as instantaneous increase in pressure) forcing exposed nuclei at mutual distances of 1 fm against their repulsive Coulomb forces, at which occurrence the strongly attractive nuclear force is activated between the two nuclei and their fusion is inevitable under the principles of ICNF reviewed below. The reader should note that ICNF (1) verifies all conceivably possible nuclear and other laws.

As described in detail in Ref. [3], ICNF (1a) was achieved via a schedule 80 carbon steel hadronic reactor of $1ft \times 2ft$ (see Figure 1) filled up with the hadronic fuel given by pure deuterium gas at 100 *psi* (following pulling out of a vacuum) that was traversed by a DC electric arc between commercially available graphite electrodes powered by a 50 kW DC-AC converter built by the U. S. company Miller Electric. The test had to be systematically interrupted following a maximum of 2 min operation to prevent melt-down of the equipment. Independent chemical analyses, done by the *Oneida ORS Laboratories* on samples of the interior gas before and after the activation of the arc, measured a macroscopic percentage of nitrogen after the activation of the arc that did not exist before, thus establishing its synthesis. The nitrogen synthesis so detected was independently confirmed by the heat produced that was definitely bigger than that provided by the 50 kW AC-DC converter.

ICNF (1a) was selected among a variety of possibilities to prevent wasteful academic discussions on the excess heat in the event interior combustion had been allowed. In fact, the interior gas, that was confirmed as being 99.99 % pure deuterium, positively cannot experience any combustion when traversed by a DC arc. Therefore, the heat measured in excess of the heat produced by the arc can solely be explained, on serious scientific grounds, as originating from ICNF (1).

ICNF (1a) was also selected among a considerable variety of possibilities to prevent wasteful academic discussions on the absence of harmful radiations. In fact, we have the synthesis of a light, natural and stable element, the nitrogen, from two ligher, natural and stable elements, the deuterium and the carbon, Therefore, when synthesis (1) occurs, there is no possibility whatsoever, not even remote, to produce harmful radiations or release radioactive waste as routinely expected by the community in nuclear fusions. In the event syntheses (1a) do not occur, there is equally the impossibility of producing harmful radiations or releasing radioactive waste because the energy of the 50 kW AC-DC converter is about one billion times short of the

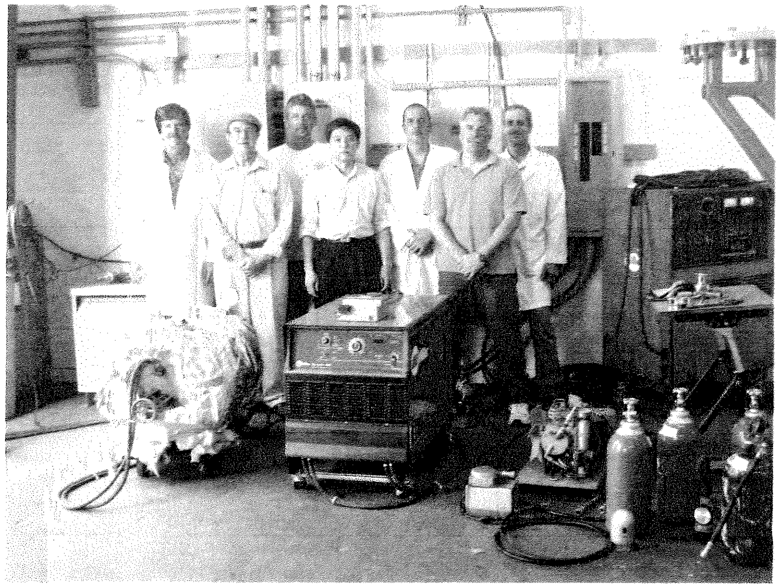

Figure 2: A view of the participants in the verification [4], showing from the left: G. West (IBR), R. M. Santilli (IBR), T. Kuliczkowski (PGTI), L. Ying (PGTI), M. Rodriguez (IBR), R. Brenna (PGTI), and C. Lynch (IBR). The picture also shows the used equipment.

energy needed to fracture the deuterium and/or the carbon nuclei for the production of the harmful radiation and waste expected by the physics community in the field.

Following the appearance of paper [3], the author requested nuclear physicists **Robert Brenna, Theodore Kuliczkowski and Leong Ying** of *Princeton Gamma Tech Instruments* to conduct independent verifications or dismissals of the results presented in Ref. [3]. Following extensive and detailed tests via the use of the same equipment and same set up of tests [3], the indicated nuclear physicists released paper [4] (see also ref. [5]) confirming all main results of Ref. [3], including: the synthesis of nitrogen from deuterium and carbon; the excess heat over that produced by the AC-DC converters; and the complete absence of harmful radiations or radioactive waste.

Refs. [3,4,5] have essentially confirmed the following *Santilli's Principles of ICNF* (see Refs. [2] for extensive studies):

PRINCIPLE 1: Need to achieve a controlled exposure of nuclei. Nuclei are naturally protected by their electron clouds, as well known. Consequently, no nuclear fusion is conceivably possible or otherwise plausible without the systematic exposure of nuclei as an evident necessary preparatory step for their fusion. This is the reason the author dedicated decades of research for the *new chemical species of Santilli magnecules* (see

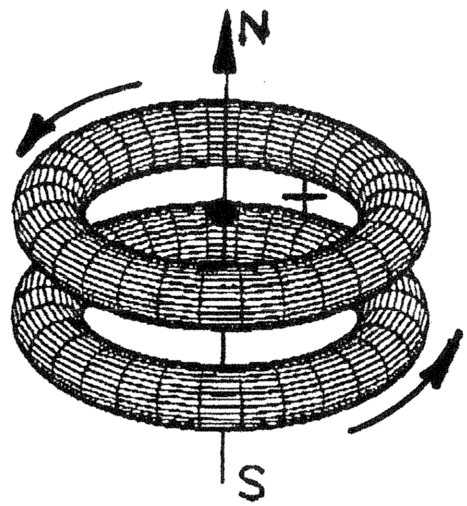

Figure 3: A conceptual view of the simplest possible example of the new chemical species of Santilli magnecule which is a necessary prerequisite for all ICNF studied in this paper.

the review in Ref. [1] or Vol. IV of Refs. [2] and original literature quoted therein). This new species is schematically represented in Figure 3 for the simplest possible biatomic case, and clearly shows the controlled exposure of nuclei via the polarization of the orbitals into toroids permitted by DC electric arc. The same picture shows the maintenance of said polarization via couplings. In the author's opinion, *the most important scientific contribution by R. Brenna, T. Kuliczkowski and L. Ying in Refs [4] has been the experimental confirmation of the existence of Santilli magnecules, not only for their evident independent chemical value, but also as a necessary prerequisite for fusion.*

PRINCIPLE 2: The need to achieve the correct spin coupling. Following the exposure of nuclei, no controlled fusion is conceivably possible, or otherwise plausible, without the additional systematic control of spin couplings. In fact, triplet couplings of spin notoriously cause strong repulsive forces in which case fusions can at best be at random. Ref. [3] established the second necessary condition for truly controlled fusions, the achievement of *systematic spin couplings either of planar singlet or of axial triplet type.* Another illustration of the fundamental character of Santilli magnecules for ICNF is visually represented in Figure 3 with the automatic achievement of the axial triplet coupling of nuclear spins (same spin direction for nuclei along the same symmetry axis).

PRINCIPLE 3: Use the minimal possible energy required by conservation laws, called "threshold energy." A reason stressed by the author for the inability by hot fusions to achieve systematic and controlled nuclear fusions (following half a century of research and the expenditure of over one billion dollars) is the use of excessive energies under which the control of the fusion is practically impossible due to inevitable instabilities and to the extreme technological difficulties for their control. Similarly, the author has stressed that a reason for the inability by cold fusions to achieve systematic and controlled fusions has been the use of insufficient energies, e.g., as needed for a systematic exposure of nuclei. These two opposite extremes illustrate the third principle of ICNF according to which, in order to avoid uncontrollable instabilities, following the achievement of the configuration of Figure 3 via the implementation of Principles 1 and 2, the fusion reactor must operate at "threshold energy," namely, the minimal possible energy needed to push the two nuclei at a mutual distance of 1 fm against their repulsive Coulomb forces, with the consequential activation of nuclear forces, at which activation fusion is simply unavoidable under the indicated premises.

2. Review of the New Tests

In this paper, we report three tests providing additional experimental confirmation of the preceding results [3,4,5]. It should be stressed to prevent misconceptions, that as it was the case for the preceding tests, the sole objective at this time of the tests reported below is that of **confirming the "existence" of systematic and controlled nuclear fusions without harmful radiation or waste.** Any expectation of "measurements" of heat produced, flow, temperature gradient and other data would be grossly premature at this time since the equipment could only be operated for a few minutes due to excessive production of heat. Also, the achievement of measurements will require the investment of millions of dollars for the construction of a hadronic reactor suitable to operate for the sufficient long time needed for meaningful measurements. Under these understandings, the new tests can be reported as follows:

TEST 1.

the main objective of this test was the experimental confirmation of the existence of the following new ICNF

$$O(18,8,0^+,17.9991) + C(12,6,0^+,12.0000) + TR \rightarrow$$
$$\rightarrow Si(30,14,0^+,29.9737) + \Delta E, \qquad (2a)$$
$$\Delta E = 0.0254\ u, \qquad (2b)$$

that also verifies all possible nuclear laws. The test was suggested by the fact that, during the years of experimentation on ICNF, the author has systematically seen a

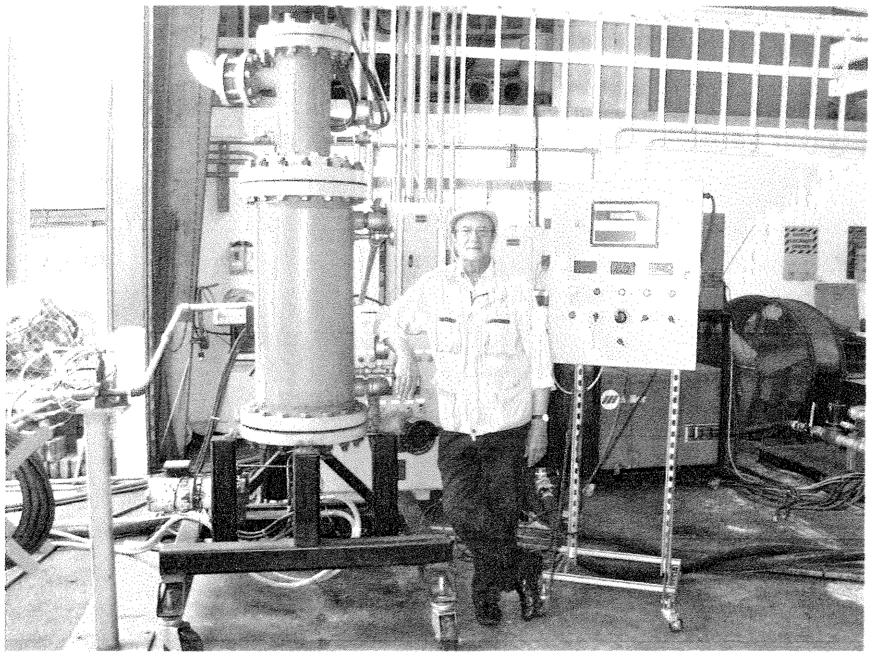

Figure 4: A picture of the hadronic reactor used in Tests 1, 2, 3.

"whitish powder" on the edge of carbon electrodes that is somewhat suggestive of the synthesis of silica.

For the test of ICNF (2a), the author and his technicians **Chris Lynch, Michael Rodriguez, Gene West, Donald Roch, Ray Jones and Jim Alban** constructed in early 2010 a new, hadronic reactor with automatic controls of the arc and main functions. as depicted in Figures 4, 5 and 6. This is the first automatic hadronic reactor for ICNF since it creates and controls automatically the DC arc, but also monitors all main features, including power, temperature, pressure, flow, trigger, and other features with automatic shut off in the event of any malfunction. The reactor essentially consists in an internal, carbon steel, schedule 80, cylindrical vessel $1\ ft \times 5\ ft$ filled up with the desired gaseous hadronic fuel and traversed by a DC arc between carbon electrodes. The internal chamber is then completed with an external water jacket used to cool down the reactor and for the production of steam. An AC-DC converter was used with 100 kW maximal power, although actual uses were restricted to 50 kW for safety. The reactor is then completed with a variety of sensors for internal as well as external temperature, pressure and other data connected to the automatic controls.

Following over one year of tests, verifications and tuning to assure the proper operation and safety of the reactor, on April 11, 2011, with the assistance of the

Figure 5: A view of the production of steam during test 3.

above indicated technicians, the author pulled a vacuum from the interior chamber of the reactor, that was subsequently filled up with commercially available oxygen at 100 *psi* pressure. The reactor was then operated for six minutes, at which time there was a violent increase in the production of steam out of the cooling jacket (see Figure 5) that forced the shut down of the reactor for safety.

After cooling off, the reactor was open and solid samples of the electrodes were sent for independent chemical analysis by *Princeton Gamma Tech Instruments* on a comparative basis with a solid sample of the same electrodes before the activation of the arc. **These analyses, entirely reported in Ref. [6], establish the distinct detection of silica following the activation of the DC arc that, under the above conditions, confirm the synthesis in laboratory of silica via ICNF (2a).** Note that no sample of the interior gas was taken because its analysis would have no impact on the desired verification, the latter dealing with a solid.

We should add that, as it was the case for all preceding tests, no measurable radiation was detected in the outside and no radioactive waste was detected in the inside of the hadronic reactor following its opening up after cooling. The various detectors used for radiations have been described in detail in Refs. [3,4] and their identification is ignored hereinafter to avoid repetitions.

TEST 2.

The controlled fusion of oxygen and carbon into silica was done because particularly important for environmental reasons since it is the premise for *the use of the green house gas CO_2 as a hadronic fuel for the production of clean energy.* In fact, a hadronic reactor can be filled up with CO_2 at pressure; the DC arc will be quite efficient in its separation into oxygen and carbon; part of the separated oxygen and carbon will evidently combust and produce CO that, in the presence of oxygen and an arc, reproduce again CO_2, thus recovering in great part the energy used for the separation of CO_2. However, jointly with the conventional combustion at a loss for the energy balance, the hadronic reactor will produce a net positive energy output due to the fusion of oxygen and carbon into silica. Test 1 described above and the second test here considered confirm the possible use of CO_2 as hadronic fuel for the production of energy without harmful radiation or waste via the indicated processes.

However, the use of oxygen in a hadronic reactor is very dangerous because it is known that virtually all substances, including metals, ignite when exposed to oxygen at high temperature. In fact, the local temperature at the tip of the DC arc when hitting the cathode is estimated as being, locally, of the order of $10^6 C$. Even though such a temperature decreases quite rapidly with the distance from the arc, it nevertheless causes a rapid increase in the temperature of the oxygen. This essentially implies the achievement of high oxygen temperatures in a matter of minutes at 100 *psi* pressure, and in seconds at higher pressures, at which value combustion of most substances exposed to oxygen is expected.

Following the adoption of due safety precautions, and in view of the indicated environmental relevance, the author and his technicians repeated Test 1 on April 14, 2011 for the specific intent of verifying or disproving results [6]. This second test was done under exactly the same conditions and setting of Test 1, thus without any modifications, to prevent variations. As predicted from carbon powder accumulated in the preceding Test 1, the internal oxygen achieved metal combustion temperature in about *three seconds* of operations, at which time an external metal fitting measuring pressure ignited and the operation has to be instantly interrupted. Nevertheless, despite its shortness, the test was sufficient to secure sample of "glassy-type small droplets" formed in the top of the cathode that were sent to *Princeton Gamma Tech Instruments* for study. **The resulting analyses, reported in full in Ref. [7], confirmed for the second time the synthesis of silica from oxygen and carbon via ICNF (2a)** via a comparison of the solid samples of Test 2 with those of the electrodes prior to the activation of the arc.

We should add again that, as it was the case for all preceding tests, no measurable radiation was detected in the outside and no radioactive waste was detected in the inside of the hadronic reactor following its opening up after cooling.

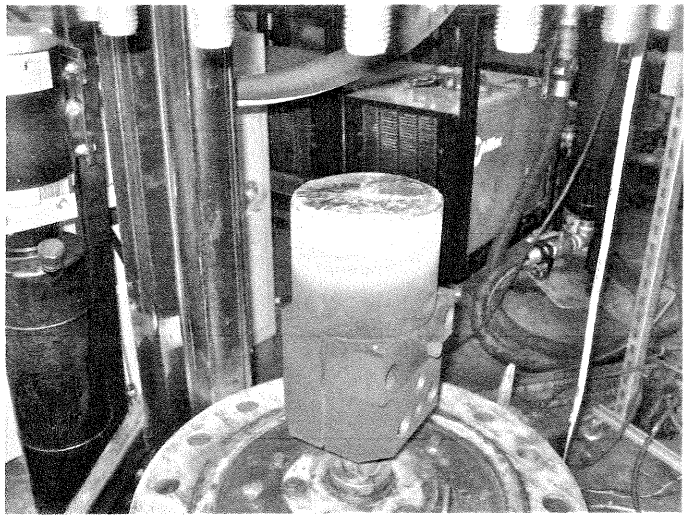

Figure 6: A view of the scorched carbon cathode following test 3.

TEST 3.
Following the successful synthesis of silica and its confirmation, among a variety of possible additional syntheses, the author selected Test 3 the ICNF of helium and carbon into the oxygen according to the rules

$$C(12,6,0^{+},12.0000) + He(4,2,0^{+},4.0026) + TR \rightarrow$$

$$\rightarrow O(16,8,0^{+}15.9949) + \Delta E \qquad (3a)$$

$$\Delta E = 0.0077\ u \qquad (3b)$$

which synthesis also verifies all possible nuclear laws.

The test was done by the author and the above identified technicians on April 15, 2011, along lines similar to the preceding ones. The interior of the reactor was cleaned, and various components replaced; a vacuum was pulled out of the interior chamber; the reactor was filled up with commercial grade helium at 100 *psi*; a sample of the interior gas was taken following due flushing and marked $He1$; the reactor was activated for about six minutes and then shut off because of excessive increase of the produced steam from the water jacket; a sample of the interior gas was then taken and, again after flushing, marked $He2$; and the two samples $He1$, $He2$ were sent to the *Oneida ORS Laboratories* for chemical analyses. **the results, reproduced in full in Ref. [8] with main results reported in Figure 7, confirm the synthesis of helium and carbon according to ICNF (3)** because, as one can see, the oxygen content decreased from 117 *ppmv* in $He1$ to a non-detectable amount in $He2$ but the CO increased from a non-detectable amount in $He1$ to 4.24% in $He2$, an increase solely possible from the synthesis of oxygen in the interior of the reactor.

We should indicate that, following test 3, samples of the electrodes were sent to *Princeton Gamma Tech Instruments* for comparative analysis with the sample electrode not exposed to the arc. The analysis was done because, following the test, the top of the cathode acquired a "glassy-type" appearance suggesting the possible synthesis of silica following that of the oxygen as per Tests 1 and 2. The results of the analyses, reported in full in Ref. [9] , show *complete absence of silica in Test 3,* and the production instead of a large peak of Fluorite that could originate from the melting of some internal plastic component of the hadronic reactor. Jointly we also note the increase of CO_2 from non-detectable in $He1$ to 914 *ppmv* in $He2$.

The latter negative result establishes that *the double nuclear synthesis, first of helium and carbon into oxygen and then of oxygen and carbon into silica, "cannot" be controlled.* In fact, during the first step, the oxygen is synthesized at the tip of the DC arc when hitting the carbon in the cathode surface. The ensuing large local production of heat as per value (3b) rapidly expels the synthesized oxygen from the DC arc, thus preventing any additional nuclear synthesis. The creation of CO is then consequential due to the great affinity of carbon and oxygen which is at the foundation of our lives.

Needless to say, the peak reported in analyses [9] for $F(19, 9, 1/2^+)$ could have interpretation other than the aboce indicated melt down of internal plastic components of the reactor, such as the ICNF of $O(18, 8, 0^+)$ and $H(1, 1, 1/2^+)$. Similarly, inspection of analyses [8] reveals the increase of the percentage of a number of elements. Of course, these increases are expected from the heat produced by the arc and the consequential conventional release of gases from the various substances composing the hadronic reactor, although some of the new elements could be the result, at least in part, of additional ICNF. The study of these possibilities requires additional tests with related analyses and they are planned for release in future presentation.

We should add again that, as it was the case for all preceding tests, no measurable radiation was detected in the outside and no radioactive waste was detected in the inside of the hadronic reactor following its opening up after cooling.

3. Concluding Remarks.

The preceding tests [3,4,5] and the additional tests presented in this paper have completed the author's intent Phase I consisting in establishing the "existence" of ICNF without harmful radiations or waste, and provided the necessary credibility for the transition to Phase II consisting in the construction of a prototype hadronic reactor producing clean electric energy in excess of that used.

Despite these promising results, the author would like to caution the reader against easy expectations of rapid achievement of Phase III, consisting in commercially avail-

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ORS REPORT NO. : 189920-001
DATE TESTED : 4/8/2011
QUANTITY TESTED : 2
PACKAGE TYPE : CYLINDER
MFG. CODE :

PO: 724
Rel. No:

SAMPLE ID		HE1	HE2						
INLET PRESSURE	torr	387	474						
NITROGEN	ppmv	665	5,431						
OXYGEN	ppmv	117	ND						
ARGON	ppmv	ND	40						
CO2	ppmv	ND	914						
MOISTURE	ppmv	1,281	3,061						
HYDROGEN	%v	0.03	3.06						
METHANE	ppmv	ND	ND						
AMMONIA	ppmv	ND	ND						
HELIUM	%v	99.8	91.7						
FLUORO-CARBONS	ppmv	ND	ND						
KRYPTON	ppmv	ND	ND						
BENZENE	ppmv	ND	158						
CO	%v	ND	4.24						

COMMENTS:

ND = None Detected
1% = 10,000 ppm

Tested per ORS SOP MEL-1070: Gas Analysis of Sealing Chamber Atmosphere.

Page : 1of1

APPROVED BY: Daniel J. Rossiter

Figure 7: A reproduction of the main results of the chemical analyses on gases for Test 3 conducted by Oneida ORS Laboratories

able new clean energies, due to the complexity of the engineering problems to be solved for extended use, as well as the large investments needed for their achievement.

Acknowledgment

The content of this note is the output of long and solitary consideration by the author expressed in ref. [2]. The main point of this note was then first discussed during the recent *Third International Conference on the Lie-Admissible Treatment of Irreversible Processes* held at the University of Kathmandu, Nepal, from January 5 to 9, 2011. The author would like to thank all participants for invaluable comments. Additionally, very special thanks are due to R. Brenna, T. Kuliczkowski and L. Ying of *Princeton Gamma Tech Instruments* and to D. J. Rossiter of *Oneida ORS Laboratories* because, without their detailed independent analyses, this paper would not have been possible. Additional thanks are due to Dorte Zuckerman for linguistic control and to Christian Corda, the Editor of the proceedings for an impeccable editorial control.

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THE NEUTRON MODEL

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Abstract

In a presence of magnetic force, the orbiting electron is found to come to a closer equilibrium state that is different from the ground state predicted by Bohr model. This distance equals to the known classical (electromagnetic) radius of the electron. This is realized when hydrogen atom is under high compression. This is achievable due to gravity supporting stars. Under this situation the gravitational and electromagnetic forces are of the same orders of magnitude. This bound minimum state is modeled for the neutron. The magnetic field resulting from the orbital motion of the electron in this minimum state is 6.06×10^{11}T. This explains the origin of the magnetic field in the neutron star.

Keywords: Nuclear structure models and methods; Neutron model; Neutron star; Nuclear physics; Strong gravity

1. Introduction

Bohr model of hydrogen relies on equality of centripetal and electric force between the electron and the proton. With other Bohr postulates, Bohr was successful in describing the spectral emission of hydrogen. However, the correct theory for hydrogen atom is the Schrodinger quantum theory. The ground state of hydrogen atom is when the electron is at Bohr radius. Bohr did not consider the magnetic field the electron experience as seen from the proton frame of reference. The inclusion of the magnetic force in the orbital motion of the electron is found to guarantee that the minimum distance the electron can exist from the nucleus is equal to the electron classical radius. In this context, Sachs considered neutron as proton bound to an electron in the nucleon domain [1]. Under this condition, the gravitational force is tantamount to electric force. The gravitational and electromagnetic radius of the electron are the same. In the ordinary atom, the gravitational force can be ignored in comparison to electric force. This is supported by the calculation we have found for this hypothesis. These data also conform with a recent study of a cosmic quantum model of the universe [2, 3, 4, 5]. In this model, the gravitational force inside the system is so huge and inseparable from the electromagnetic forces.

In 1920, Ernest Rutherford hypothesized the possible existence of the neutron as due to the disparity found between the atomic number of an atom and its atomic mass which could be explained by the existence of a neutrally charged particle within the nucleus [6].

The magnetic field created at the electron site due to the proton electric field (E), in the rest frame of the proton, is given by Biot-Savart law, viz., [7]

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}\,, \qquad B = \frac{kev}{r^2 c^2}\,. \tag{1}$$

This magnetic field gives rise to the spin-orbit interaction between the electron spin and the orbital angular momentum. This is well corrected by Thomas factor as [8]

$$\Delta E = \frac{1}{m^2c^2}\frac{k}{r^3}\,\vec{L}\cdot\vec{S}\,, \qquad \vec{\mu}_s = \frac{e}{m}\vec{S}\,, \qquad k = \frac{1}{4\pi\varepsilon_0}\,. \tag{2}$$

In the next section, we will show that this magnetic field is responsible for preventing the electron to fall into the nucleus (proton). The stationary orbit is given by

$$\frac{mv^2}{r} = \frac{ke^2}{r^2} = eE\,. \tag{3}$$

However, when an electron (magnetic moment) is placed in a magnetic field, the electron will precess with Larmor frequency, ω_L, due to its spin. This frequency is defined as [8]

$$\omega_L = \frac{e}{2m}B\,. \tag{4}$$

2. The classical electron radius

The force on the electron in hydrogen atom is governed by the Lorentz force

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}\,. \tag{5}$$

Using eqs. (1) & (3), the centripetal force will become

$$\frac{mv^2}{r} = \frac{ke^2}{r^2} + \frac{ke^2v^2}{r^2c^2}\,. \tag{6}$$

The above equation can be written as

$$r^2 - \frac{L^2}{kme^2}\,r + \frac{L^2}{m^2c^2} = 0\,. \tag{7}$$

This is solved to give

$$r_1 = \frac{k\,e^2}{mc^2}\,, \tag{8}$$

as the minimum distance from the nucleus, which is the electron classical radius (r_e), and

$$r_2 = \frac{L^2}{kme^2} - r_1\,, \tag{9}$$

which is the new Bohr orbit with magnetic field effect included.

Equation (7) can be written as

$$r_1 + r_2 = \frac{L^2}{kme^2}\,. \tag{10}$$

The first term represents the electron orbit radius when only electric force is present. It is thus interesting to note that the effect of including the internal magnetic field in hydrogen in electron orbital motion is to shorten the orbit by the electron classical radius.

In gravitation, one can define a similar radius (the gravitational radius), that any orbiting body of mass m around a central mass M, as

$$r_g = \frac{GM}{c^2}\,, \tag{11}$$

as a consequence of gravitomagnetic force. This is half the Schwarzschild's radius of a black hole. Thus, eq.(8) defines the electromagnetic (Schwartzchild-like) radius. It is the distance that when reached by a body, the body can'not escape out of the electric force exerted on.

According to the general theory of relativity, the equation of the orbit of an object around a black hole is given by [11]

$$r^2 - \frac{L^2}{GMm^2}\,r + \frac{3L^2}{m^2c^2} = 0\,.$$

Comparing the above equation with eq.(7) reveals that an electric black hole may also exist that is analogous to gravitational one. This implies that the shortest distance is $r_0 = 3r_g$.

The Bohr orbit, velocity and energy of the electron are given by

$$r_n = \left(\frac{\hbar^2}{mke^2}\right)n^2\,, \qquad v_n = \left(\frac{ke^2}{\hbar}\right)\frac{1}{n}\,, \qquad E_n = -\frac{mk^2e^4}{2\hbar^2n^2}\,. \tag{12}$$

It is interesting to see that the classical electron radius, i.e., r_1, corresponds to Bohr orbit with $n = \alpha = \frac{ke^2}{\hbar c} = \frac{1}{137}$, so that its velocity will be the speed of light in vacuum. In this case, the magnetic and electric forces are equal. This critical case defines a quantum case in which the electron and the proton become so close and may oscillate with high frequency. We call this orbit the "quantum orbit". The energy of the electron in this case will be

$$E_0 = -\frac{1}{2}\,mc^2 \tag{13}$$

representing the quantum fluctuation energy of an orbiting electron around the nucleus. It can also be related to the zero energy due to matter contributions. This may be linked to the minimum energy of a harmonic oscillator, viz., $E_0 = \frac{1}{2}\hbar\,\omega$. If we, thus, add the matter contribution and the field contribution (oscillator), we get a vanishing total zero energy, $E_{min} = 0$. This concept will resolve the cosmological constant problem of the standard cosmology.

3. High pressure

The electron exists in the quantum orbit when a high pressure is exerted on hydrogen. Such a high pressure exists in neutron stars. In a presence of huge magnetic field, the electron can exist in the quantum orbit defined above. We can assume it circulates the proton with speed of light (at most). Its mass will increase and be relativistic (m_*). Recall that in Dirac formulation the electron speed is c too. The proton-neutron mass difference can be obtained from the orbital energy of the electron in eq.(13) by assigning the right value for m. In beta decay, the protonium gives a proton, an electron and the binding energy of the electron converted into neutrino.

An external magnetic field that needed to set the electron in circular motion with radius r_e is when

$$evB = \frac{mv^2}{r_e}\,, \qquad v \cong c\,. \tag{14}$$

This approximation is applicable under extreme conditions of high pressure. Using eq.(8), the above equation yields

$$B_c = \frac{mc}{r_e e} = \frac{m^2 c^3}{k\,e^3} = 6.06 \times 10^{11}\ \mathrm{T}\,. \tag{15}$$

This is a typical magnetic field found in neutron stars. Thus, this huge magnetic field may be generated as a result of the electron motion around the proton. Hence, neutron stars are not composed of neutrons, but a hydrogen system of electron-proton minimum states (*protonium*). The protonium state acts as an unbalanced dipole, since the charge is distributed over two unequal sizes. This asymmetry may explain the anomalous magnetic moment of the neutron. If this state represents a neutron, then the mass difference between the proton and the neutron is carried by the electron orbital energy, as evident from eq.(13).

One can estimate the pressure exerted on these protoniums as

$$P_q = \frac{F_q}{A} = \frac{ke^2/r_e^2}{4\pi r_e^2} = \frac{m^4c^8}{4\pi k^3 e^6} = 2.93 \times 10^{29}\ \mathrm{N/m^2}\,. \qquad (16)$$

This is in fact a typical nuclear pressure. It is also equal to the pressure at the center of a neutron star. One can also measure the current resulting from the circulating motion of the electron in protonium as

$$I_q = \frac{ec}{2\pi r_e} = \frac{mc^3}{2\pi ke} = 2.7 \times 10^3\ \mathrm{A} \qquad (17)$$

and the resulting magnetic field is that of a loop, i.e.,

$$B_q = \frac{\mu_0 I_q}{2\pi r_e} = \frac{\mu_0 m^2 c^5}{4\pi k^2 e^3} = \frac{m^2c^3}{ke^3}\,, \qquad \text{where} \qquad \frac{\mu_0 c^2}{4\pi} = k\,. \qquad (18)$$

It is interesting that the two magnetic fields (eq.(15) and (18)) are the same. The magnetic flux is defined by

$$\phi_q = B_q A = 4\pi \frac{ke}{c} = \frac{e}{\varepsilon_0 c} = 6.03 \times 10^{-17}\mathrm{Wb}.$$

Comparing eq.(15) and (18) shows clearly that the magnetic field of the neutron star is that due to the motion of the electrons in the protonium states.
The capacitance and inductance of the protonium are defined by

$$L_q = \frac{\phi_q}{I_q} = \frac{\mu_0 e^2}{2mc^2} = 1.96\times 10^{-31}\,\mathrm{H}\,, C_q = \frac{e}{ke/r_e} = \frac{e^2}{mc^2} = 3.12\times 10^{-25}\,\mathrm{F}\,.$$

One can define the impedance of the protonium as

$$Z_q = \frac{\tau_q}{C_q} = \frac{1}{\varepsilon_0 c} = 120\,\pi = 377\,\Omega\,,$$

where τ_q is the time constant ($\tau_q = 2T_q$, see eq.(25)). It can alternatively be defined as $Z_q = \sqrt{\frac{L_q}{C_q}}$. This is the same as the impedance of the free space (vacuum). Thus, despite the huge electric and magnetic field inside the protonium, yet the space inside is that of a true vacuum.

Let us now consider the magnetic moment due to the electron in the protonium state. This is given by

$$\mu_q = -\left(\frac{e}{2m}\right) L = -\frac{e}{2m}(\hbar\alpha) = -\frac{k\,e^3}{2mc} \quad = -6.75\times 10^{-26}\ \text{J/T} \quad (19)$$

This can be compared with the measured value of the neutron and protons magnetic moments, viz. $\mu_n = -9.66\times 10^{-27}$ J/T and $\mu_p = 1.41\times 10^{-26}$ J/T. We can get the exact value by making some linear combinations of eq.(19) and μ_p.

One can further find the electric field due to protonium state as

$$E_q = \frac{ke}{r_e^2} = \frac{m^2c^4}{ke^3} = 1.8\times 10^{20}\ \text{V/m}\,. \quad (20)$$

The electromagnetic energy density (or pressure) inside the protonium is

$$u_q = \frac{1}{2}\epsilon_0 E_q^2 + \frac{B_q^2}{2\mu_0} = \frac{m^4c^8}{4\pi k^3e^6} = 2.93\times 10^{29}\ \text{J/m}^3\,. \quad (21)$$

This is the same as the eq.(16).

The centripetal acceleration of the electron in the protonium state is

$$a_q = \frac{v^2}{r_e} = \frac{2mc^4}{ke^2} = 6.4\times 10^{31}\ \text{m/s}^2\,. \quad (22)$$

The centripetal force of the electron is

$$F_c = \frac{mv^2}{r_e} = \frac{2m^2c^4}{ke^2}\,. \quad (23)$$

If we consider the protonium as a diploe, we can calculate the power radiated due to Larmor dipole radiation as

$$\mathcal{P}_q = \frac{2}{3}\frac{ke^2a_q^2}{c^3} = \frac{2}{3}\frac{m^2c^5}{ke^2} = 5.8\times 10^{9}\ \text{W}\,. \quad (24)$$

The time for the electron to circulate the proton in the protonium is

$$T_q = \frac{2\pi r_e}{v} = \frac{2\pi ke^2}{mc^3} = 5.9\times 10^{-23}\ \text{s}\,. \quad (25)$$

This is a typical nuclear time. If we define the power as

$$\mathcal{P} = E_0/T_q = \frac{1}{4\pi}\frac{m^2c^5}{ke^2}\,, \tag{26}$$

which apart from the prefactor of 4π, is the same as that of eq.(24). The reaction force of the emitted radiation , the Abraham-Lorentz force, is given by

$$F_q = \frac{\mu_0 e^2}{6\pi c}\frac{da}{dt} = \frac{m^2c^4}{3\pi ke^2} = 1.48\times 10^{-3}\ \mathrm{m/s^2}\,, \tag{27}$$

which apart from the prefactor of 3π, is not very different from the centripetal force in eq.(23).

The Larmor frequency for protonium as defined in eq.(4) will be, using eqs.(1) & (8),

$$\omega_q = \frac{mc^3}{2ke^2}\,, \tag{28}$$

while the gravitational Larmor precession is [9]

$$\omega_g = \frac{GMv}{2r^2c^2}\,, \tag{29}$$

of an orbiting body about a central mass, M. The latter being independent of the mass of the orbiting body.

4. Cosmic quantum mechanics

In a recent study, we have constructed the physical quantities characterizing the physical world at all scales [2]. We employ only the four fundamental constants, c, k, G, and $\hbar$. The above physical quantities for the protonium state conform with the quantities obtained from this study. From an earlier work, we have shown that inside the nuclear region, the Newton's constant (G_N) is given by [3, 4, 5]

$$G_N \sim 10^{40}G\,. \tag{30}$$

In the nuclear domain, the magnetic moment and current are given by

$$\mu_{\mathrm{N}} = \left(\frac{G_N\hbar^2}{k}\right)^{\frac{1}{2}}\,, \tag{31}$$

and

$$I_{\mathrm{N}} = \left(\frac{c^6}{G_N k}\right)^{\frac{1}{2}} . \tag{32}$$

The nuclear electric field intensity is given by

$$E_{\mathrm{N}} = \left(\frac{c^7 k}{\hbar G_N^2}\right)^{\frac{1}{2}} , \tag{33}$$

The nuclear magnetic field density is

$$B_{\mathrm{N}} = \left(\frac{c^5 k}{\hbar G_N^2}\right)^{\frac{1}{2}} . \tag{34}$$

The nuclear magnetic flux density is given by

$$\Phi_{\mathrm{N}} = \left(\frac{k\hbar}{c}\right)^{\frac{1}{2}} . \tag{35}$$

The magnetic (electric) field contribution to mass density is given by

$$\rho_{\mathrm{mN}} = \left(\frac{B^2}{k}\right) . \tag{36}$$

The pressure exerted by nuclear medium(quantum) is given by

$$P_N = \left(\frac{c^7}{G_N^2 \hbar}\right) . \tag{37}$$

The acceleration of the quantum fluid filling the space-time inside the nucleus is given by

$$a_N = \left(\frac{c^7}{G_N \hbar}\right)^{\frac{1}{2}} . \tag{38}$$

The amount of energy emitted per unit time per unit area (energy flux) in the nuclear region is given by

$$\Sigma_N = \left(\frac{c^8}{G_N^2 \hbar}\right) . \tag{39}$$

These quantities are the same to the corresponding ones in the previous section if we set, $\frac{G_N m_g^2}{\hbar c} = \alpha^2$, where, $m_g = \sqrt{m_e m_p}$, is the geometric mean of electron and proton masses. This implies that the electric and the gravitational forces are of the same magnitude. It is interesting to consider the case when $n = \sqrt{\alpha}$, that yields, $r_n = \frac{\hbar}{mc}$ and $E_n = -\frac{1}{2}mc^2\alpha$.

The intensity in eq.(39) can be obtained from eq.(24) or (26) as

$$\Sigma_q = \frac{\mathcal{P}}{4\pi r_e^2} = \frac{1}{6\pi}\left(\frac{m^4c^9}{k^3e^6}\right), \tag{40}$$

which has the same order of magnitude. Under these severe conditions the electron in the atom will be in the protonium state. The gravitational force is so immense as indicated above. Thus, the electron does not combine with proton to form the neutron. It exists in the protonium state at a distance equals to the electron classical radius from the proton. If we had substituted the relativistic mass for the electron, eqs.(8) - (28), will change by some orders of magnitude. This can be done to give eq.(13) the required value (the neutron electron mass difference) or the required value for the neutron magnetic moment in eq.(19).

It is remarkable that $\omega_q = \omega_g$ for protonium state only if $G \to 10^{40}G$, as evident from eqs.(28) and (29). This makes the description of protonium state electromagnetically is equivalent to the gravitational description. This is the only state where gravity is of great importance.

5. Neutron star

Stars (mainly hydrogen) are affected by their self-gravitating energy and the gas energy. At equilibrium (balance) the gravity and gas (neutron Fermi gas) pressures are equal. According to Newtonian gravity, this occurs when the gas radius (R) is related to the star mass M by [10]

$$R = \frac{3}{8}\left(\frac{3}{2\pi^4}\right)^{1/3}\left(\frac{h^2}{Gm_n^{8/3}}\right)M^{-1/3}. \tag{41}$$

The gravitational pressure is defined as

$$P_g = \frac{3}{5}\left(\frac{GM^2}{4\pi R^4}\right) . \tag{42}$$

Owing to eq.(41) and (42), a neutron star of mass $M = 1.5M_{\odot}$ will produce a radius of $10.75\,km$ and a pressure of 2×10^{33} Pa. In our present model, a pressure of $\sim 10^{33}$ Pa will produce a magnetic field of 10^{12}T. This is also apparent from substituting eqs.(16) and (18) to yield

$$P_q = \left(\frac{4\pi}{\mu_0}\right) B_q^2 . \tag{43}$$

6. Conclusions

The gravitational interaction has been assumed to be negligible inside the hydrogen atom. However, at nuclear scales the gravitational force could be enormously large. When the magnetic force is considered in Bohr model, this force prevents the electron from falling into the nucleus (proton). The the other effect can be enormously large. We have found that the correct electron orbit due to the presence of the internal magnetic field is to shorten the electron orbit by a distance that equals to the electron classical radius. The effect of a subminimum (quantum) state is the creation of a protonium state that is a neutral state. The impedance of this state is equal to that of free space. This may be interpreted as the space inside this state is the same as that of vacuum. The physical properties of this state coincide with the neutron, but the protonium has asymmetric charge distribution. Moreover, our findings agree with the data obtained from recent hypothesis of a cosmic quantum mechanics picture of the universe. In this hypothesis, the gravitational force in the nuclear domain is as great as 10^{40} as the ordinary gravity. This makes gravity inseparable from electromagnetism.

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THE UNIFIED QUANTUM WAVE EQUATIONS

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Abstract

The quaterionic formulation of quantum mechanics yields the unified quantum wave equation (UQWEs). From these equations, Dirac, Klein - Gordon and Schrodinger equations can be derived. While the UQWEs represent a matter wave (de Broglie), the Maxwell equations represent a transverse wave (field). Owing to UQWEs, the spin-0 and spin-1/2 particle are described by a wavepacket consisting of waves traveling to the left and to the right with speed of light. UQWEs show that spin-0 and spin-1/2 are in continuous states of creation and annihilation that are compatible with Heisenberg uncertainty relation. The creation - annihilation process is a result of the time translation property of the particle wavefunction. These are $E' = E - im_0c^2$ and $E' = E \pm m_0c^2$, for Klein-Gordon' and Dirac' particles, respectively. It is found that $\frac{\hbar}{m_0c^2}$ is the period of the creation -annihilation process.

Keywords: Quantum mechanics; Unified quantum models; Dirac equation; Klein-Grdon equation; Telegraph equation

1 Introduction

Quantum mechanics has been developed by Schrodinger who employed de Broglie hypothesis about the matter wave. The solution of Schrodinger is a plane wave. However, a particle should be represented by a collection of waves (wavepacket). Schrodinger equation is valid for a particle moving at non-relativistic speed. Later, Klein and Gordon developed a quantum theory based on Einstein's relativity. The resulting equation represents the motion of spin -0 massive particles. However, the probability density obtained from this equation is not positive definite. This urges physicists to replace it by a more convenient equation. For this Dirac developed his quantum theory. To remedy this impasse, Dirac presented a first order differential equation in space and time.

Maxwell had used the quaternions formulation to write his electromagnetic equations. The resulting number of equations are too many (20 equations). These equations look absurd. Only after Gibbs and Heaviside invented the vector analysis, Maxwell equations, in their present form, became conspicuous. Since that time quaternions had been absent from physics except for some recent limited trials. The surprising work has come recently when I reintroduce quaternions in quantum mechanics employing new ideas [1]. As a result, unified quantum wave equation is obtained. From this equation we have derived the Dirac, Klein-Gordon and Schrodinger equations [2]. The UQWEs express Dirac's equation as a second-order wave equation [1, 3]. Consequently, Klein-Gordon as well as Dirac equations are of the same mathematical structure. With the aid of Arbab-Widatallah complex transformations, Dirac equation in its new form is derived from the UQWEs using the complex mass transformation $m_0 \to i\beta m_0$ [4, 5]. Moreover, Dirac and Klein-Gordon equations are found to stem from a massless wave equation upon making the mass translations (shift/rotation). Like Maxwell equations, UQWEs involve scalar and vector waves. The scalar wave represents longitudinal wave while vectorial wave represents transverse waves. The vectorial wave can be some sort of spin (or polarization) wave that some magnetic systems have exhibited. It can also be related to the field associated with the particle (electric, magnetic, gravitational,...

etc). In a recent work, we have shown that for spin-0 particles, the vector field is related to its acceleration [6]. In our present case, one finds ($\varphi \Leftrightarrow \psi_0$) and ($\vec{A} \Leftrightarrow \vec{\psi}$) as fundamental fields. Aharonov and Bohm demonstrated that the real fields that the quantum nature of the particle reveals are the vector potential $\vec{A}$ and scalar potential φ [7]. The phenomenon in which an electron is affected by $\vec{A}$ and φ produces the interference pattern of wavefunction is called Aharonov-Bohm effect. This is confirmed experimentally by setting up an environment with $\vec{A}$ and φ, while having zero electric and magnetic fields. This poses the question whether $\vec{A}$ and φ are more fundamental than $\vec{E}$ and $\vec{B}$. Moreover, $\vec{A}$ can be decomposed ($\vec{A}_\perp$) in transverse and parallel ($\vec{A}_{||}$) components. In 1930 Fermi showed that $\vec{A}_{||}$ and φ give rise to the instantaneous Coulomb interactions between the charged particles, whereas $\vec{A}_\perp$ accounts for the electromagnetic radiation of charged moving particles.

2 Universal quantum wave equation

We have recently derived a system of unified quantum wave equations [1, 2]

$$\vec{\nabla}\cdot\vec{\psi} - \frac{1}{c^2}\frac{\partial\psi_0}{\partial t} - \frac{m_0}{\hbar}\,\psi_0 = 0\,, \tag{1}$$

$$\vec{\nabla}\psi_0 - \frac{\partial\vec{\psi}}{\partial t} - \frac{m_0c^2}{\hbar}\,\vec{\psi} = 0\,, \tag{2}$$

and

$$\vec{\nabla}\times\vec{\psi} = 0\,. \tag{3}$$

Equations (1) - (3) can be solved to give

$$\frac{1}{c^2}\frac{\partial^2\psi_0}{\partial t^2} - \nabla^2\psi_0 + 2\left(\frac{m_0}{\hbar}\right)\frac{\partial\psi_0}{\partial t} + \left(\frac{m_0c}{\hbar}\right)^2\psi_0 = 0\,, \tag{4}$$

and

$$\frac{1}{c^2}\frac{\partial^2\vec{\psi}}{\partial t^2} - \nabla^2\vec{\psi} + 2\left(\frac{m_0}{\hbar}\right)\frac{\partial\vec{\psi}}{\partial t} + \left(\frac{m_0c}{\hbar}\right)^2\vec{\psi} = 0\,. \tag{5}$$

This is a dissipative wave equation for spin - 0 particle. It is a generic Telegraphy equation representing signal transmission. Such a wave arises when friction or other dissipative force produces a damping (proportional to the velocity of vibration), whose effect in the wave equation is the inclusion of the term proportional to $\frac{\partial\psi}{\partial t}$.

The solution of eq.(4)/or eq.(5) is of the form

$$\psi_0(x,t) = A\exp(-\frac{m_0c^2}{\hbar}t)\exp(\pm 2\pi\, i(ckt - \vec{k}\cdot\vec{r}))\,,$$

where A = const. and $\vec{k}$ is the propagation constant. It represents an undistorted damped wavepacket moving to the left and right.

It is interesting to note that eqs.(4) and (5) represent a scalar wave (longitudinal) and a vector wave that are concomitant with the particle motion. The vector wave is only a feature of our present equation and does not exist in Schrodinger, Dirac or Klein-Gordon description. Therefore, other physical properties can be associated with this vector nature of the particle. Hence, the complete physical description of the particle will be performed in terms of these two waves.

Let us now write

$$\psi_0(r,t) = \exp\,(-\frac{m_0c^2}{\hbar}\,t)\,\varphi(r,t)\,. \tag{6}$$

Substitute eq.(6) in eq.(4) to get

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = 0\,. \tag{7}$$

Hence, $\varphi(r,t)$ satisfies the wave equation. Therefore, the UQWE is a relativistic equation. It is remarkable that the UQWEs describe the particle by a scalar and vector quantities. Thus, the full description of the particle motion can be made using these two quantities only. Moreover, since eqs.(5) & (6) are of a Telegraph-type equation that represents the motion of the electric signal in a wire, then the motion of a particle in space mimics signal propagation. Moreover, eq.(5 and (6) are of special nature that describes the propagation of an undistorted signal along the wire. Hence, the particle of spin-0 travels in space undistortedly.

Dirac's equation can be written as [8]

$$\frac{1}{c}\frac{\partial\psi}{\partial t} + \vec{\alpha}\cdot\vec{\nabla}\psi + \frac{im_0c\,\beta}{\hbar}\,\psi = 0\,. \tag{8}$$

where $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\alpha^2 = \beta^2 = 1$ and $\vec{\sigma}$ are the Pauli matrices. Equation (8) can be written as

$$\frac{1}{c}\frac{\partial \psi}{\partial t} + \frac{i\, m_0 c\, \beta}{\hbar}\psi = -\vec{\alpha} \cdot \vec{\nabla}\psi\,. \tag{9}$$

Squaring the two sides of eq.(9) yields

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{i\, m_0 c\, \beta}{\hbar}\right)^2 \psi = \left(-\vec{\alpha} \cdot \vec{\nabla}\right)^2 \psi\,. \tag{10}$$

Since $\alpha^2 = \beta^2 = 1$, eq.(10) yields

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + 2\left(\frac{i\, m_0 \beta}{\hbar}\right)\frac{\partial \psi}{\partial t} - \left(\frac{m_0 c}{\hbar}\right)^2 \psi = 0\,. \tag{11}$$

It is remarkable to know that eq.(11) can be obtained directly from eq.(4) if we let

$$m_0 \to i\beta\, m_0\,, \tag{12}$$

in eq.(4). This entitles the wavefunction ψ to be represented by a two-component construct, viz., $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$.

Let us now write

$$\psi(r,t) = \exp\left(-\frac{i\, m_0 c^2 \beta}{\hbar}\, t\right) \chi(r,t)\,. \tag{13}$$

Substituting eq.(13) in eq.(11) yields

$$\frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} - \nabla^2 \chi = 0\,. \tag{14}$$

Once again, under the energy translation of the particle's wavefunction both Dirac and Klein-Gordon equations describe a massless particle. This is the equation of a massless particle (wave).

Let us now write the wavefunction

$$\psi_0(r,t) = \exp\left(-\vec{\kappa} \cdot \vec{r}\right) \varphi(r,t)\,, \qquad \kappa = \text{const.}\,, \tag{15}$$

and substitute it in eq.(4) to get

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \frac{m_0 c^2}{\hbar}\right)^2 \varphi - \left(\vec{\nabla} - \vec{\kappa}\right)^2 \varphi = 0\,, \tag{16}$$

or

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial\eta^2} - \nabla'^2\varphi = 0\,, \tag{17}$$

where

$$\frac{\partial}{\partial\eta} = \frac{\partial}{\partial t} + \frac{m_0c^2}{\hbar}\,, \qquad \vec{\nabla}' = \vec{\nabla} - \vec{\kappa}\,. \tag{18}$$

Equations (17) and (18) are the wave equation for a massless particle interacting with an external field described by κ. This can be compared with a massless particle interacting with electromagnetic potentials $\vec{A}$ and φ where $\vec{A} \equiv -\frac{i\hbar}{e}\,\vec{\kappa}$ and $V \equiv i\,m_0c^2$.

Similarly, let us now write the wavefunction

$$\psi(r,t) = \exp\left(-\vec{\kappa}\cdot\vec{r}\right)\chi(r,t)\,, \qquad \kappa = \text{const.}\,, \tag{19}$$

and substitute it in eq.(11) to get

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + i\frac{m_0c^2}{\hbar}\right)^2\chi - \left(\vec{\nabla} - \vec{\kappa}\right)^2\chi = 0\,, \tag{20}$$

or

$$\frac{1}{c^2}\frac{\partial^2\chi}{\partial\tau^2} - \nabla'^2\chi = 0\,, \tag{21}$$

where

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + i\frac{m_0c^2}{\hbar}\,, \qquad \vec{\nabla}' = \vec{\nabla} - \vec{\kappa}\,. \tag{22}$$

Equation (22) can be seen as representing a massless particle interacting with an external vector field $\vec{\kappa}$ in potential energy m_0c^2. Thus, Dirac and Klein-Gordon particles, with mass m_0, are equivalent to massless particles (waves) interacting, with constant real and imaginary scalar potential with the same constant vector potential, respectively. These correspond to $E'_{KG} = E + im_0c^2$, $E'_D = E + m_0c^2$, and $\vec{p}\,' = \vec{p} + i\,\hbar\vec{\kappa}$, for Klein-Gordon (KG) and Dirac (D) particles, respectively.

Consequently, eqs.(15) and (19) can be seen as representing the local gauge transformation of the Klein-Gordon and Dirac wavefuncions, χ and φ, respectively. In this case the four vector potential $A_\mu = (A_0\,, \vec{A})$ will be $(m_0c^2\,, -\frac{i\hbar}{e}\,\vec{\kappa})$ for Dirac and $(im_0c^2\,, -\frac{i\hbar}{e}\,\vec{\kappa})$ for Klein-Gordon. In electromagnetism, the gauge transformation is obtained via $A'_\mu = A_\mu + \partial_\mu\lambda$. This corresponds, in our present theory, to $\lambda = -\frac{i\hbar}{e}\vec{\kappa}\cdot\vec{r}$. This analogy is quite interesting. The terms A_μ and $\partial_\mu\lambda$ can be seen as describing the transverse an paralel components of the vector potential, respectively. We have shown recently that the parallel component of $\vec{A}$, i.e., $\partial_\mu\lambda$, gives rise to a longitudinal wave [6].

The dispersion relation arising from eq.(4) is

$$\omega_\pm = \frac{m_0c^2}{\hbar}i \pm ck\,, \tag{23}$$

so that the group velocity is

$$v_g = \frac{\partial\omega}{\partial k} = \pm c\,. \tag{24}$$

3 Creation and annihilation of particles

It interesting that the mass of the particle disappears in eq.(14) while appears in eq.(11). Hence, at a time τ (after an interval of $\frac{\hbar}{m_0c^2}$ the particle loses its mass (annihilates) and then again being created after the same time. Therefore, as time goes on the particle experiences continuously a processes of creation and annihilation. This process is governed by the time of uncertainty owing to the Heisenberg uncertainty relation ($\Delta t\,\Delta E \geq \hbar$).

Under the transformation

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + i\frac{m_0c^2}{\hbar}\beta\,, \tag{25}$$

eq.(11) can be written as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial\tau^2} - \nabla^2\psi = 0\,. \tag{26}$$

And under the transformation

$$\frac{\partial}{\partial\eta} = \frac{\partial}{\partial t} + \frac{m_0c^2}{\hbar}\,, \tag{27}$$

eq.(4) becomes

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial\eta^2} - \nabla^2\psi = 0\,. \tag{28}$$

Using eq.(8), Dirac equation, eq.(11), can be written as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\nabla^2\psi-2\left(\frac{m_0 c\, i}{\hbar}\right)\beta\vec{\alpha}\cdot\vec{\nabla}\psi+\left(\frac{m_0 c}{\hbar}\right)^2\psi=0\,, \tag{29}$$

which under the transformation

$$\vec{\nabla}\,'=\vec{\nabla}+i\frac{m_0 c\beta}{\hbar}\,\vec{\alpha}\,, \tag{30}$$

becomes

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t2}-\nabla'^2\psi=0\,. \tag{31}$$

Once again, this is a wave equation of a massless particle. Thus, a particle of spin-0 or spin-1/2 undergoes a process of creation and annihilation during its propagation in space and time.

Hence, eqs.(15) & (16) and eqs.(17) & (18) are compatible with eqs.(6) & (14). Therefore, while Dirac's particle undergoes a virtual process of creation and annihilation, Klein-Gordon's particle undergoes the same process in real time. It is interesting to observe that eqs.(7) and (18) are connected by the energy translation, viz., $E\,'=E-i\,m_0c^2$ and $E\,'=E\pm m_0c^2$, to eqs.(6) and (13), respectively. Hence, a massless wave equation can be obtained from Dirac equation by employing the energy translation (shift), $E\,'=E\pm m_0c^2$, instead of setting the particle mass to zero, and vice versa. These may be attributed to time advance and time retard translation. Similarly, Klein-Gordon equation can be obtained from UQWE by employing the energy translation (rotation), $E\,'=E-i\,m_0c^2$. This may be attributed to time rotation of the wavefunction. Imaginary mass is like imaginary frequency, designates a dissipation in the oscillating system. Moreover, these transformations are equivalent to set, $\omega\,'=\omega-i\,\omega_c$, and $\omega\,'=\omega\pm\,\omega_c$, where $\omega_c=\frac{m_0c^2}{\hbar}$.

The wavefunction of Dirac particle is a wavepacket consisting of waves traveling to the right and left with speed of light in opposite direction. The dimension of this wavepacket is $L=\frac{\hbar}{m_0c}$. The transformation in eq.(30) tells us that the creation and annihilation processes occur periodically over space and time. One can argue that the spin of the Dirac particle is due to the rotation of the two waves comprising the particle around each other. For an electron, each wave (identity) has a mass half that of the electron, i.e., $m_0/2$. The spin angular momentum will thus be

$$S = I\omega = \Sigma_i m_i r_i^2 \omega_i = m_1 r_1^2 \omega_1 + m_2 r_2^2 \omega_2 \,, \omega_1 = \omega_2 = \frac{c}{r} \,. \qquad (32)$$

Hence,

$$S = \frac{1}{2}\hbar \,, \qquad r = L/2. \qquad (33)$$

This coincides with the quantum value predicted by Dirac. It has long been believed that the spin is not a classical concept! It thus becomes obvious that the electron spin is analogous to the angular momentum of the classical circularly polarized wave. It has been shown by Belinfante in 1939, that it is possible that the electron spin can be considered as an angular momentum, generated by the circulation of the energy flow in the field of electron wave [9]. It is of importance to remark that the property in eq.(30) is not applicable to eq.(4) of spin -0 particles.

It is interesting to notice that because of the damping nature of the the spin-0 zero particles (eq.(6)), their interactions are of short range. This is in agreement with Yukawa's theory. This is unlike the interactions of the spin -1/2 particles which have oscillatory wave nature, as evident from eq.(14). Owing to this property, spin-1/2 particles have long range interactions. The conversion of mass into energy and ice versa are a manifestation of Einstein's mass-energy equivalence, since both equations emerge from this equation.

4 Conclusions

We have developed unified quantum mechanics wave equations that have an analogy with Maxwell equations, and yield Dirac and Klein - Gordon equations. These unified equations represent, like Maxwell equations, scalar (longitudinal) and vectorial (transverse) waves. In the context of these equations, the solution of the modified Klein-Gordon equation is that of the normal Klein-Gordon equation where the frequency will be $\omega' = \omega - i\,\omega_c$. However, the solution of the Dirac equation is obtained from the wavefunction of the standard wave equation by allowing the frequency of the wave to be $\omega' = \omega \pm \omega_c$. Moreover, the Dirac and Klein-Gordon equations are found to be equivalent to an equation of massless particle interacting with constant vector and scalar potentials. The transformed wavefunction for Dirac and Klein-Gordon equation are that of the local gauge transformation with a linear gauge function. Dirac and Klein-Gordon equations are shown to exhibit creation - annihilation process consistent with Heisenberg uncertainty equation.

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QUANTUM MECHANICS, MATHEMATICAL STATISTICS AND DEFORMED COMMUTATION RELATIONS

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Abstract

In this paper, quantum mechanics and its relation to mathematical statistics are investigated. The Fisher information, wave function, kinetic energy, two new uncertainty relations, Klein-Gordon equation, probability density current and deformed commutation relations are discussed.

Keywords: Fisher information, wave function, kinetic energy, uncertainty relations, Klein-Gordon equation, probability density current, deformed commutation relations.

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1 Fisher information

Measurement in quantum mechanics is described in a statistical way. For this reason, we discuss in this paper mutual relation of the formalism of quantum mechanics and mathematical statistics.

First, we discuss the Fisher information — a very important quantity appearing in mathematical statistics. In the most simple form, it can be introduced as follows (see e.g. [1-18]).

We start with the normalization condition for the probability density $\rho(x) = |\psi(x)|^2$, where ψ is the wave function

$$\int \rho \, \mathrm{d}x = 1.$$

Here, integration is performed from the minus infinity to plus infinity. For the sake of simplicity, we assume also that ρ has the property

$$\lim_{x \to \pm\infty} x^n \rho = 0, \quad n = 0, 1, 2. \tag{1}$$

Therefore, we limit ourselves to discussion of the so-called bound states.

Now, we perform integration by parts in the normalization condition and get

$$[(x-a)\rho]_{x=-\infty}^{\infty} - \int (x-a) \frac{\partial \rho}{\partial x} \, \mathrm{d}x = 1,$$

where a is an arbitrary real number. Taking into account condition (1) we get the starting point of the following discussion

$$\int (x-a) \frac{\partial \rho}{\partial x} \, \mathrm{d}x = -1.$$

Further, we make use of the Schwarz inequality for the inner product $\langle u, v \rangle = \int u^* v \, \mathrm{d}x$ of two complex functions u and v

$$\langle u, u \rangle \langle v, v \rangle \geq |\langle u, v \rangle|^2. \tag{2}$$

Putting

$$u = (x-a)\sqrt{\rho}, \quad v = \frac{1}{\sqrt{\rho}} \frac{\partial \rho}{\partial x}$$

and using Schwarz inequality (2) we get

$$\int (x-a)^2 \rho \, \mathrm{d}x \int \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 \mathrm{d}x \geq 1,$$

where the second integral is called the Fisher information I

$$I = \int \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 \mathrm{d}x.$$

This inequality is usually written in the form

$$\int (x-a)^2 \rho \, \mathrm{d}x \; I \geq 1. \tag{3}$$

This result is very general and does not depend on the concrete meaning of the variable x.

Interpretation of the last inequality is similar to that of the uncertainty relations in quantum mechanics since for given I the integral $\int (x-a)^2 \rho \, \mathrm{d}x$ cannot be smaller than $1/I$ and vice versa. The minimum of the integral $\int (x-a)^2 \rho \, \mathrm{d}x$ is obtained for $a = \int x\rho \, \mathrm{d}x$. In a more general form, it is is possible to derive the so-called Rao–Cramér inequalities [19-21].

2 Wave function

The wave function ψ can always be written in the form

$$\psi = \mathrm{e}^{(is_1 - s_2)/\hbar}, \tag{4}$$

where s_1 and s_2 are real functions and $\hbar$ is the Planck constant. It follows from here that

$$\rho = |\psi|^2 = \mathrm{e}^{-2s_2/\hbar}.$$

Therefore, the Fisher information can be written in the equivalent form [7-11, 13, 17, 18]

$$I = \int \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 \mathrm{d}x = \frac{4}{\hbar^2}\int \left(\frac{\partial s_2}{\partial x}\right)^2 \mathrm{e}^{-2s_2/\hbar} \mathrm{d}x.$$

3 Kinetic energy

Using Eq. (4) for the wave function we can write the quantum-mechanical kinetic energy T in the form [7-11, 13, 17, 18]

$$T = \int \frac{(\partial s_1/\partial x)^2 + (\partial s_2/\partial x)^2}{2m} \mathrm{e}^{-2s_2/\hbar} \mathrm{d}x.$$

Therefore, kinetic energy can be written as a sum of two terms

$$T = T_1 + T_2,$$

where

$$T_1 = \int \frac{(\partial s_1/\partial x)^2}{2m} \mathrm{e}^{-2s_2/\hbar} \mathrm{d}x, \quad T_2 = \frac{\hbar^2 I}{8m}.$$

The first part of the kinetic T_1 is analogous to the classical kinetic energy given by the expression $T_{class} = (\nabla S)^2/(2m)$ known from the Hamilton-Jacobi theory.

The second part of the kinetic energy

$$T_2 = \frac{\hbar^2 I}{8m}$$

is proportional to the Fisher information I. Due to T_2, the kinetic energy of the bound states cannot equal zero. Therefore, the Fisher information plays very important role in quantum mechanics.

4 Heisenberg uncertainty relations

As above, we write the wave function ψ in form (4). The Heisenberg uncertainty relation for the coordinate x and momentum p has the form [22]

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4},$$

where

$$\langle(\Delta x)^2\rangle = \int (x - \langle x\rangle)^2 |\psi|^2 \mathrm{d}x, \quad \langle(\Delta p)^2\rangle = \int |(\hat{p} - \langle\hat{p}\rangle)\psi|^2 \mathrm{d}x,$$

$\hat{p} = -i\hbar(\partial/\partial x)$ and $\langle\,\rangle$ denotes the usual quantum-mechanical mean value.

Analogously to the kinetic energy, $\langle(\Delta p)^2\rangle$ can be split into two parts [7-10, 13, 15-18]

$$\langle(\Delta p)^2\rangle = \langle(\Delta p_1)^2\rangle + \langle(\Delta p_2)^2\rangle,$$

where

$$\langle(\Delta p_1)^2\rangle = \int \left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)^2 \mathrm{e}^{-2s_2/\hbar}\mathrm{d}x$$

and

$$\langle(\Delta p_2)^2\rangle = \int \left(\frac{\partial s_2}{\partial x}\right)^2 \mathrm{e}^{-2s_2/\hbar}\mathrm{d}x.$$

Similarly to the first part of the kinetic energy T_1, $\langle(\Delta p_1)^2\rangle$ can be interpreted within generalization of classical mechanics in which the classical momentum $p = \partial S/\partial x$, where S is the classical action, is replaced by $\partial s_1/\partial x$ and the probability density $\rho = |\psi|^2 = \mathrm{e}^{-2s_2/\hbar}$ is introduced.

The second part

$$\langle(\Delta p_2)^2\rangle = \int \left(\frac{\partial s_2}{\partial x}\right)^2 \mathrm{e}^{-2s_2/\hbar}\mathrm{d}x$$

is, analogously to T_2, proportional to the Fisher information I

$$I = \int \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 \mathrm{d}x = \frac{4}{\hbar^2}\int \left(\frac{\partial s_2}{\partial x}\right)^2 \mathrm{e}^{-2s_2/\hbar}\mathrm{d}x = \frac{4}{\hbar^2}\langle(\Delta p_2)^2\rangle.$$

For $\langle(\Delta p_1)^2\rangle = 0$ (for example for real wave functions), the Heisenberg uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4},$$

is equivalent to inequality (3) with $a = \langle x\rangle$.

5 Klein-Gordon equation

In physics, we have to take into account not only the probability density ρ but also the probability density current $\mathbf{j}$ describing the motion in space. For this reason, we introduce generalized spatial and time Fisher informations I''_x and I''_t [13, 17, 18]

$$I''_x = \frac{4}{\hbar^2}\int_{t=0}^{\infty}\int\left[\left(\frac{\partial s_1}{\partial x}\right)^2 + \left(\frac{\partial s_2}{\partial x}\right)^2\right]\mathrm{e}^{-2s_2/\hbar}\mathrm{d}x\mathrm{d}t = \int_{t=0}^{\infty}\int\left|\frac{\partial \psi}{\partial x}\right|^2\mathrm{d}x\mathrm{d}t$$

and

$$I''_t = \frac{4}{\hbar^2}\int_{t=0}^{\infty}\int\left[\left(\frac{\partial s_1}{\partial t}\right)^2 + \left(\frac{\partial s_2}{\partial t}\right)^2\right]\mathrm{e}^{-2s_2/\hbar}\mathrm{d}x\mathrm{d}t = \int_{t=0}^{\infty}\int\left|\frac{\partial \psi}{\partial t}\right|^2\mathrm{d}x\mathrm{d}t.$$

Since there are no potentials in the last two Fisher informations, they correspond to a free particle.

To describe physical phenomena in a way independent of the choice of the concrete inertial system, we require that the combined space-time Fisher information equals a real constant K independent of the state of the investigated system

$$\frac{I''_t}{c^2} \pm I''_x = K,$$

where c is the speed of light and the sign in front of the spatial Fisher information I''_x can be either $+$ or $-$. By considering two cases of a particle in rest and a particle with very large kinetic energy it can be shown that [13, 17, 18]

$$K \geq 0$$

and the minus sign in the last equation has to be taken

$$\frac{I''_t}{c^2} - I''_x = K.$$

In this way, the correct signs of the metric of the special relativity and the relativistic invariance of the theory is obtained.

The last equation can be then written in the form

$$\int_{t=0}^{\infty}\int\left(\frac{1}{c^2}\left|\frac{\partial \psi}{\partial t}\right|^2 - \left|\frac{\partial \psi}{\partial x}\right|^2 - \frac{\hbar^2 K}{4}|\psi|^2\right)\mathrm{d}x\mathrm{d}t = 0.$$

This functional must be independent of ψ

$$\int_{t=0}^{\infty}\int\left(\frac{1}{c^2}\frac{\partial\delta\psi^*}{\partial t}\frac{\partial\psi}{\partial t}-\frac{\partial\delta\psi^*}{\partial x}\frac{\partial\psi}{\partial x}-\frac{\hbar^2K}{4}\delta\psi^*\psi\right)\mathrm{d}x\mathrm{d}t+c.c.=0,$$

where δ denotes the variation. Performing integration by parts with respect to t in the first term and with respect to x in the second one and assuming that variations $\delta\psi$ and $\delta\psi^*$ equal zero at the borders of the integration region we have

$$\int_{t=0}^{\infty}\int\delta\psi^*\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\hbar^2K}{4}\right)\psi\,\mathrm{d}x\mathrm{d}t+c.c.=0.$$

The condition that this equation has to be fulfilled for arbitrary values of $\delta\psi$ and $\delta\psi^*$ yields the equation of motion

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\hbar^2K}{4}\right)\psi=0.$$

Introducing the rest mass m_0

$$K=\frac{4m_0^2c^2}{\hbar^4}$$

and generalizing to three dimensions we obtain the well-known Klein-Gordon equation

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{m_0^2c^2}{\hbar^2}\right)\psi=0.$$

Dirac equation and many other equations of motion of physics can be obtained in a similar way [3-6, 13, 17, 18].

6 Two new uncertainty relations

Now we show that the Heisenberg uncertainty relation can be replaced by two stronger uncertainty relations [13, 15-18, 23].

First, we take

$$u = \Delta x \sqrt{\rho}, \quad v = \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) \sqrt{\rho}.$$

Then, the Schwarz inequality yields the first uncertainty relation

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_1)^2 \rangle \geq \left[\int \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) \mathrm{e}^{-2s_2/\hbar} \mathrm{d}x \right]^2. \tag{5}$$

Here, the function $\partial s_1/\partial x$ corresponds to the classical momentum $\partial S/\partial x$ and the relation has the usual meaning known from mathematical statistics: the product of variances of two quantities is greater than or equal to the square of their covariance. Depending on the functions s_1 and s_2, the square of the covariance of the coordinate and momentum at the right-hand side can have arbitrary values greater than or equal to zero.

The second uncertainty relation can be obtained in an analogous way for

$$u = \Delta x \sqrt{\rho}, \quad v = \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle \right) \sqrt{\rho}$$

with the result

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_2)^2 \rangle \geq \left[\int (x - \langle x \rangle) \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle \right) \mathrm{e}^{-2s_2/\hbar} \mathrm{d}x \right]^2.$$

The right-hand side of this relation can be simplified

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_2)^2 \rangle \geq \frac{\hbar^2}{4}. \tag{6}$$

This uncertainty relation follows from the Schwarz inequality in a similar way as the first one, however, the covariance $\langle u, v \rangle$ is in this case constant and equals $\hbar/2 > 0$ independently of the concrete form of the function s_2. We note also that this relation is for $\langle x \rangle = a$ equivalent to inequality (3) for the Fisher information.

We see that the Heisenberg uncertainty relation can be replaced by two more detailed uncertainty relations. Uncertainty relation (5) can be understood as the standard statistical inequality between the coordinate x and momentum represented by the function $p = \partial s_1/\partial x$. Uncertainty relation (6) can be understood as the standard statistical inequality, too. However, because of the specific form of the covariance $\langle u, v \rangle$ which equals $\hbar/2$ independently of the function s_2, the left-hand side of this relation must be greater than or equal to $\hbar^2/4$.

We note that the sum of uncertainty relations (5) and (6) is equivalent to the so-called Robertson-Schrödinger relation for the coordinate and momentum. The Heisenberg uncertainty relation can be obtained from the sum of the uncertainty relations by neglecting the first term on its right-hand side. Therefore, two new uncertainty relations are stronger than the corresponding Heisenberg and Robertson-Schrödinger uncertainty relations [24-26].

For general discussion of this approach see [7-10].

7 Example: Free particle

We assume that the wave function of a free particle is at time $t = 0$ described by the gaussian wave packet [17, 18, 23]

$$\psi(x, 0) = \frac{1}{\sqrt{a\sqrt{\pi}}} \mathrm{e}^{-x^2/(2a^2)+ikx}$$

with the energy

$$E = \frac{\hbar^2}{4ma^2} + \frac{\hbar^2 k^2}{2m},$$

where $a > 0$ and k are real constants. By solving the time Schrödinger equation we get

$$\psi(x, t) = \frac{1}{\sqrt{a\sqrt{\pi}}} \frac{\sqrt{1 - \frac{i\hbar t}{ma^2}}}{\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}}$$

$$\times \exp\left\{ -\frac{\left(x - \frac{\hbar k}{m} t\right)^2}{2a^2 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} + i \left[\frac{kx + \frac{\hbar t x^2}{2ma^4} - \frac{\hbar k^2}{2m} t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2}\right] \right\}.$$

The corresponding functions s_1 and s_2 equal

$$s_1(x,t) = \hbar k \frac{x + \frac{\hbar t x^2}{2ma^4 k} - \frac{\hbar k}{2m} t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2} - \hbar \arctan \frac{\hbar t}{ma^2},$$

$$s_2(x,t) = \frac{\hbar}{2} \left\{ \frac{\left(x - \frac{\hbar k}{m} t\right)^2}{a^2 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} - \ln \frac{1}{a\sqrt{\pi}\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}} \right\}.$$

As it could be anticipated, the mean momentum and the mean coordinate have the form

$$\langle \hat{p} \rangle = \left\langle \frac{\partial s_1}{\partial x} \right\rangle = \hbar k, \quad \langle x \rangle = \frac{\hbar k}{m} t.$$

The mean square deviations of the coordinate and momentum are given by the equations

$$\langle (\Delta x)^2 \rangle = \frac{a^2}{2} \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right], \quad \langle (\Delta p_1)^2 \rangle = \frac{\hbar^4 t^2}{2m^2 a^6 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]}$$

and

$$\langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{2a^2 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]}.$$

The left–hand side and the right-hand side of uncertainty relation (5) have the same value

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_1)^2 \rangle = \left\langle \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) \right\rangle^2 = \frac{\hbar^4 t^2}{4m^2 a^4}.$$

Therefore, the first uncertainty relation (5) is fulfilled with the equality sign.

Calculating the left–hand side of uncertainty relation (6) we obtain

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{4}$$

and see that the second uncertainty relation (6) is fulfilled with the equality sign, too.

8 Equality sign

The equality sign in uncertainty relations (5) and (6) is obtained if the functions s_1 and s_2 are quadratic functions of x of the form $p(t)x^2 + q(t)x + r(t)$, where real coefficients $p(t)$, $q(t)$ and $r(t)$ can depend on time [17, 18, 23]. All functions s_1 and s_2 given in our example fulfill this condition.

It is worth to notice that this condition for the first uncertainty relation is independent of the form of the function s_1. Therefore, the equality sign in this relation can be achieved for much larger class of the wave functions than in case of the Heisenberg or Robertson–Schrödinger uncertainty relations. It is interesting not only from theoretical but also from the experimental point of view.

9 Standard commutation relations

Now we return back to the normalization condition for the wave function

$$\int |\psi|^2 \mathrm{d}x = 1.$$

Performing integration by parts and assuming $x|\psi|^2 \to 0$ for $x \to \pm\infty$ we get

$$\int x \left(\frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right) \mathrm{d}x = -1.$$

Multiplying this equation by $-i$ we obtain the equation [11, 13, 17, 18, 23]

$$\int \left[(x\psi)^* \left(-i \frac{\partial \psi}{\partial x} \right) - \left(-i \frac{\partial \psi}{\partial x} \right)^* x\psi \right] \mathrm{d}x = 2i \int x \frac{\partial s_2}{\partial x} \mathrm{e}^{-2s_2} \mathrm{d}x = i.$$

The resulting equation

$$\int \left[(x\psi)^* \left(-i \frac{\partial \psi}{\partial x} \right) - \left(-i \frac{\partial \psi}{\partial x} \right)^* x\psi \right] \mathrm{d}x = i.$$

contains the operator $-i(\partial/\partial x)$ which appears here as simple mathematical consequence of integration by parts applied to the normalization condition and indicates validity of a more general operator equality

$$[x, -i(\partial/\partial x)] = i.$$

Except for the factor $\hbar$ determining the choice of units, this commutation relation agrees with the commutation relation

$$[x, \hat{p}] = i\hbar.$$

between the coordinate x and momentum operator $\hat{p} = -i\hbar(\partial/\partial x)$ known from quantum mechanics.

It is seen that existence of the commutation relation for the coordinate and momentum in standard quantum mechanics is closely related to the existence of the normalized probability distribution $\rho(x)$ and relation $\rho = |\psi|^2$. Similar commutation relations should appear in any statistical theory formulated analogously to that discussed above.

10 Probability density current

Now we discuss the probability density current $\mathbf{j}$ [11, 13, 17, 18, 23]. As in continuum mechanics, we assume

$$\mathbf{j} = \rho\,\mathbf{v},$$

where $\mathbf{v}$ is "velocity". We have in the Hamilton-Jacobi theory

$$\mathbf{v} = \frac{\nabla S}{m},$$

where S is the Hamilton action and m is the mass. By analogy with these expressions we can take in quantum mechanics

$$\mathbf{j} = \rho\frac{\nabla s_1}{m}.$$

Then we get

$$\mathbf{j} = \frac{\hbar}{m}\Big[-i(\sqrt{\rho}\,\mathrm{e}^{is_1/\hbar})^*\nabla(\sqrt{\rho}\,\mathrm{e}^{is_1/\hbar}) + i\nabla\rho/2\Big],$$

where $\rho = \mathrm{e}^{-2s_2/\hbar}$. Using the wave function in the form $\psi = \sqrt{\rho}\,\mathrm{e}^{is_1/\hbar}$ we obtain the well-known result

$$\mathbf{j} = \frac{\hbar}{2mi}[\psi^* \nabla\psi - \nabla\psi^* \psi].$$

11 Deformed commutation relations

Now we make an attempt to find prescription for the probability density and inner product that would lead to the deformed commutation relation in the form

$$p\,x[-i(\partial/\partial x)] - q\,[-i(\partial/\partial x)]x = i,$$

where $p > 0$ and $q > 0$ are real numbers. For the sake of simplicity, we put $\hbar = 1$ here. We assume the normalization condition for the probability density $\rho(x)$ in the usual form

$$\int \rho\,\mathrm{d}x = 1.$$

Performing integration by parts and assuming $x\rho \to 0$ for $x \to \pm\infty$ we get

$$\int x \frac{\partial \rho}{\partial x} \mathrm{d}x = -1.$$

This equation containing the first derivate with respect to x is the starting point of the following discussion.

In standard quantum mechanics, we use the relation

$$\rho = |\psi|^2.$$

Now, let us try a bit more general expression

$$\rho = |\psi|^p\,|\psi|^q.$$

Repeating similar procedure as above we get

$$\int \left[q\,x \frac{\partial|\psi|}{\partial x} - p \frac{\partial(x|\psi|)}{\partial x} \right] |\psi|^{p+q-1} \mathrm{d}x = -1.$$

This result indicates that

$$q\,x(\partial/\partial x) - p(\partial/\partial x)x = -1$$

or

$$p\,x[-i(\partial/\partial x)] - q\,[-i(\partial/\partial x)]x = i.$$

Thus, the probability density $\rho = |\psi|^p\,|\psi|^q$ leads to the deformed commutation relation $p\,x[-i(\partial/\partial x)] - q\,[-i(\partial/\partial x)]x = i$.

However, an attempt to define the corresponding inner product in the form

$$\langle\varphi, \psi\rangle = \int [\varphi^p]^* \psi^q \mathrm{d}x$$

or

$$\langle\varphi, \psi\rangle = \int \mathrm{e}^{-i\arg\varphi}|\varphi|^p \mathrm{e}^{i\arg\psi}|\psi|^q \mathrm{d}x$$

fails since these formulas do not obey the usual mathematical properties of the inner product.

It is seen that this naive approach fails and that a more systematic theory has to be used (see [27, 28] and references therein).

12 Conclusions

- Statistical description of measurement can be used as the starting point for formulating consistent physical theories. It is especially valid for quantum mechanics and quantum theory in general.

- The complex wave function ψ carries information on two real quantities: probability density $\rho = |\psi|^2$ and probability density current $\mathbf{j} = \hbar/(2mi)[\psi^*\nabla\psi - \nabla\psi^*\psi]$.

- The Fisher information depending on the form of the probability density ρ or the envelop of the wave function is an important part of the kinetic energy.

- The Fisher information appears also in the uncertainty relations.

- The Fisher information can be used to find equations of motion.
- It is possible to derive two uncertainty relations that are stronger than the Heisenberg uncertainty relation. In these relations, classical and quantum descriptions are separated.
- Standard commutation relations can be obtained from the normalization condition $\int \rho \, \mathrm{d}x = 1$.
- Our attempt to get mathematical structure of quantum mechanics with deformed commutation relations in a similar way as it can be done for standard quantum mechanics has not been succesfull. It must be done in a more systematic way as in the papers of prof. R.M. Santilli (see e.g. [27, 28]).

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