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QUANTUM OPTICS AND FUNDAMENTALS OF QUANTUM MECHANICS

Contribution to Understanding the Mathematical Structure of Quantum Mechanics[1]

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Abstract—Probabilistic description of results of measurements and its consequences for understanding quantum mechanics are discussed. It is shown that the basic mathematical structure of quantum mechanics like the probability amplitudes, the Born rule, commutation and uncertainty relations, probability density current, momentum operator, and rules for including the scalar and vector potentials and antiparticles can be obtained from the probabilistic description of results of measurement of the space coordinates and time. Equations of motion of quantum mechanics, the Klein–Gordon equation, the Schrödinger equation, and the Dirac equation are obtained from the requirement of the relativistic invariance of the space–time Fisher information. The limit case of the δ-like probability densities leads to the Hamilton–Jacobi equation of classical mechanics. Many-particle systems and the postulates of quantum mechanics are also discussed.

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1. INTRODUCTION

Quantum mechanics is one of the most completely tested physical theories (see, e.g., [1–4]). At the same time, the standard approach to introducing quantum mechanics based on the sometimes contraintuitive postulates does not clarify the roots of quantum mechanics and the exact physical meanings of the postulates and their interpretation is the subject of continuing discussion (see, e.g., [5–25]). It is not satisfactory and, in our opinion, it is necessary to concentrate on a more direct description of the probabilistic character of measurements and its consequences. Such an approach can clarify the most important assumptions made in quantum mechanics and contribute to understanding quantum mechanics as a probabilistic theory of certain class of physical phenomena.

In this paper, we do not want to develop a new interpretation of quantum mechanics. Rather, we would like to contribute to the understanding of the standard, or Copenhagen, interpretation by illuminating its basic ideas by means of the probabilistic description of measurements. It is to be noted that the approach used in this paper is different from that usually used in physics. To explain experimental results, one introduces some physical quantities and equations of motion for these quantities. Then, the consequences of these equations are investigated and compared with results of measurements and this procedure is repeated. The probabilistic character of physical phenomena in the quantum world is well-known and has to be respected in any attempt at understanding quantum mechanics. Therefore, we describe results of measurements in a probabilistic way and ask the question of what the mathematical apparatus is that can describe this situation in the simplest manner. Using such probabilistic or information theoretical approach, the basic mathematical structure of quantum mechanics, except for equations of motion, is obtained. Equations of motion are found from the requirement of the relativistic invariance of the theory. This paper is a substantially extended version of the papers [26]. A slightly different approach based on the principle of extreme physical information was used by Frieden, who derived the most important equations of motion in physics (see [15]).

In this paper, we do not discuss measurement processes in detail and assume that measuring apparatuses for measuring spatial coordinates and time exist. Models based on the probabilistic description of the measured system and measuring apparatus interacting with the thermodynamic bath can be found, for example, in [27, 28], see also [29].

Most likely, the best approach is to start with measurement of the spatial coordinates and time. In this paper, we show that the basic mathematical structure of quantum mechanics, like the probability amplitudes, the Born rule, commutation and uncertainty relations, probability density current, kinetic energy, the momentum operator, or the rules for including the scalar and vector potentials and antiparticles, can be obtained from the probabilistic description of results of measure-

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ment of the spatial coordinates and time by means of the probability density and probability density current (Sections 2–11). Equations of the motion of quantum mechanics, the Klein–Gordon equation, and the Dirac equation are obtained from the requirement of the relativistic invariance of the generalized space–time Fisher information (Section 12). The Schrödinger equation is known to be the nonrelativistic limit of the Klein–Gordon equation. The limit case of the localized probability densities yields the Hamilton–Jacobi equation of classical mechanics (Section 13). Generalization to many-particle systems is performed in Section 14. Postulates of quantum mechanics are discussed in Section 15.

2. PROBABILITY DENSITY AND THE BORN RULE

Physical experiments show that results of measurements very often have a probabilistic character, which is related to the well-known experimental conditions of measurements. The interaction of the measured system with the measuring apparatus and the rest of the world cannot, in general, be neglected, as measuring apparatuses are not described in detail but rather only on the macroscopic level, real physical detectors have limited resolution and efficiency, the experimental control of the initial conditions is limited, etc. As a result, the resolution of physical experiments is always limited and the assumption that measurements can be arbitrarily exact (made for example in classical mechanics) is not valid. Therefore, results of measurements must be described in a probabilistic manner (see, e.g., [17, 23]).

General definition of the mean value of a real physical quantity A can be written either in the continuous,

$$\langle A\rangle = \int A\rho(A)dA, \tag{1}$$

or discrete form,

$$\langle A\rangle = \sum_i A_i\rho_i. \tag{2}$$

Here, A resp. A_i denotes the continuous resp. discrete values of the quantity A that can be obtained in measurements and $\rho(A)$ resp. ρ_i are the relative weights of the corresponding probabilistic distributions.

To be more concrete, we will first discuss the measurement of the coordinate x. Results of repeated measurements of the coordinate x can be, in physically reasonable cases, characterized by the mean values of the moments

$$\langle x^n\rangle = \int x^n\rho(\mathbf{r}, t)dV, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where integration is carried out over the whole space, $dV = dxdydz$, and $\rho(\mathbf{r}, t) \geq 0$ is a normalized probability density to obtain the coordinate x in measurement made at time t,

$$\int\rho dV = \langle x^0\rangle = 1. \tag{4}$$

This normalization condition is supposed to be valid at all times t.

First, we perform the integration by parts with respect to the variable x in Eq. (3) and get

$$\int\frac{x^{n+1}}{n+1}\rho\Big|_{x=-\infty}^{\infty}dydz - \int\frac{x^{n+1}}{n+1}\frac{\partial\rho}{\partial x}dV = \langle x^n\rangle. \tag{5}$$

Assuming that the first term in this equation equals zero for physically reasonable ρ, we get

$$\int x^{n+1}\frac{\partial\rho}{\partial x}dV = -(n+1)\langle x^n\rangle, \quad n = 0, 1, 2, \ldots. \tag{6}$$

We will show that this simple result has consequences that are interesting from the point of view of the basic structure of quantum mechanics.

The last equation can be rewritten in the form of the inner product

$$(u, v) = -(n+1)\langle x^n\rangle, \tag{7}$$

defined in the usual way as

$$(u, v) = \int u^* v dV. \tag{8}$$

Here, the star denotes the complex conjugate and the functions u and v can be taken in the general form

$$u = x^{n+1}\psi, \tag{9}$$

$$v = (1/\psi^*)\partial\rho/\partial x, \tag{10}$$

where $\psi = \psi(\mathbf{r}, t)$ is an arbitrary complex function. At this point, it is sufficient to consider real functions only. However, the assumption of the complex functions makes possible the further generalization discussed in the following sections. We note that Eq. (7) has the same physical and mathematical content as Eq. (6). Our aim is to find conditions for the function ψ that will lead to the most physically reasonable and mathematically simple formulation of the theory.

A generally valid property of the inner product (8) is the Schwarz inequality,

$$(u, u)(v, v) \geq |(u, v)|^2. \tag{11}$$

Due to Eq. (7), the Schwarz inequality yields, in our case,

$$(u, u)(v, v) \geq (n+1)^2\langle x^n\rangle^2, \quad n = 0, 1, 2, \ldots, \tag{12}$$

where

$$(u, u) = \int x^{2n+2}|\psi|^2 dV, \quad n = 0, 1, 2, \ldots \tag{13}$$

and

$$(v, v) = \int \frac{1}{|\psi|^2}\left(\frac{\partial\rho}{\partial x}\right)^2 dV. \quad (14)$$

Until now, ψ could be an arbitrary complex function and the integrals (u, u) and (v, v) have, in general, no direct relation to the mean values $\langle x^n \rangle$, characterizing the measurement of the coordinate x. Therefore, inequalities (12) are for general ψ only a mathematical result without any direct physical meaning.

However, since ψ can be an arbitrary function, we can require that the integrals (u, u) in Eq. (13) do have a physical meaning and the function ψ obeys the conditions

$$(u, u) = \int x^{2n+2}|\psi|^2 dV = \int x^{2n+2}\rho dV, \quad n = 0, 1, 2, \ldots. \quad (15)$$

If these conditions are fulfilled, the inequalities (12) contain the moments $\langle x^n \rangle$ used above for describing the results of measurements. The inner product (v, v) will be discussed below.

Conditions (15), together with the normalization condition that can be applied to ψ

$$\int \rho dV = \int |\psi|^2 dV = 1 \quad (16)$$

do not determine the relation between ρ and $|\psi|^2$ uniquely. They do show, however, that the most simple, physically reasonable relation between the probability density ρ and probability amplitude ψ has the form of the Born rule as follows:

$$\rho = |\psi|^2 \quad (17)$$

or

$$\psi = \sqrt{\rho}e^{is_1}, \quad (18)$$

where $s_1 = s_1(\mathbf{r}, t)$ is a real function and its physical meaning will be discussed in Section 8. We note that the probability amplitude ψ can be multiplied by an arbitrary phase factor $\exp(i\alpha)$, where α is a real number. It follows from the last two equations that the function ψ can be called the probability amplitude.

Defining the function $s_2 = s_2(\mathbf{r}, t)$ by the equation

$$\sqrt{\rho} = e^{-s_2}, \quad (19)$$

the probability amplitude $\psi = \psi(\mathbf{r}, t)$ can be written also in the "eikonal" form

$$\psi = e^{is}, \quad (20)$$

where s_1 and s_2 are the real and imaginary parts of s

$$s = s_1 + is_2 \quad (21)$$

and the function $s_2 = -(1/2)\ln\rho$ gives the form of the probability density ρ.

We have seen that the integration by parts applied to the definition of the mean values (3) yields Eq. (6), which has interesting physical and mathematical implications. Its general consequence in form of the inequality (12) does not contain physically relevant quantities unless we assume that the function ψ obeys conditions (15). It is seen that this approach does not lead to the basic concept of quantum mechanics—the probability amplitude—uniquely. It shows, however, that the simplest solution of the conditions (15) and the normalization condition (16) has the form of the well-known Born rule (17) ([30], see also [9, 14, 31]).

It is to be noted that our way of writing the probability amplitude in Eq. (20) is very similar to the expression used by Bohm [7, 8] (see also Madelung [32]), i.e.,

$$\psi = Re^{iS/\hbar}. \quad (22)$$

To get the same formula as Bohm, we can set $\hbar = 1$, $S = s_1$, and $R = \exp(-s_2)$. For this reason, it is not surprising that some parts of the following discussion are similar to that performed by Bohm. The most important differences between our approach and that of Bohm are summarized in the Conclusions.

3. COMMUTATION RELATION

Now we return to the normalization condition

$$\int |\psi|^2 dV = 1. \quad (23)$$

Performing the integration by parts similar to that in Eq. (5) and assuming that $x|\psi|^2 \longrightarrow 0$ for $x \longrightarrow \pm\infty$, we get

$$\int x\left(\frac{\partial\psi^*}{\partial x}\psi + \psi^*\frac{\partial\psi}{\partial x}\right)dV = -1. \quad (24)$$

Multiplying this equation by $-i$, we obtain the equation

$$\int\left[(x\psi)^*\left(-i\frac{\partial\psi}{\partial x}\right) - \left(-i\frac{\partial\psi}{\partial X}\right)^* x\psi\right]dV = 2i\int x\frac{\partial s_2}{\partial x}e^{-2s_2}dV = i. \quad (25)$$

In the usual approach, the momentum operator $\hat{p}_x = -i\hbar(\partial/\partial x)$ and the corresponding Hilbert space of the wave functions are postulated. Then, the commutation relation $[x, \hat{p}_x] = i\hbar$ h appears to be a rather trivial mathematical identity. However, our approach is different. We do not postulate the form of the momentum operator here; instead, we show that Eq. (25), containing the operator $-i(\partial/\partial x)$, appears in the probabilistic description as a simple consequence of the integration by parts applied to the normalization condition (4) and the Born rule (17).

Equation (25) indicates the validity of a more general operator equality,

$$[x, -i(\partial/\partial x)] = i. \quad (26)$$

Except for the factor $\hbar$ determining the choice of units, this commutation relation agrees with the commutation relation $[x, \hat{p}_x] = i\hbar$ h between the coordinate x and the momentum operator $\hat{p}_x = -i\hbar(\partial/\partial x)$ known from quantum mechanics (for the momentum, see also Section 9).

We note also that the second integral in Eq. (25) does not depend on s_1 and shows that this equation is the relation for $s_2 = -(1/2)\ln\rho$ only. Therefore, the existence of the commutation relation (26) is related to the existence of the probability distribution ρ. In classical mechanics, where the probability distribution ρ disappears, this commutation relation disappears, too. The probability density ρ does not appear in the classical dynamics, which depends only on the function $S_1 = \hbar s_1$ playing the role of the classical action S (see Section 13).

It is seen that the usual quantization based on the transition from the classical coordinates and momentum to the coordinate and momentum operators obeying the commutation relations $[x, \hat{p}_x] = i\hbar$ corresponds to assuming the probabilistic character of measurements described by ρ. Thus, discussion in Sections 2 and 3 helps one to understand the postulates of quantum mechanics and contributes to understanding the basic ideas of quantum mechanics.

We note that similar commutation relations can be expected to be valid in any probability theory formulated analogously to that discussed above. In the limit case, when the probabilistic distribution of the results of measurements can be neglected (as, for example, in classical mechanics), such commutation relations need not be considered.

4. FIRST UNCERTAINTY RELATION

The uncertainty relation for the coordinate x and the operator $-i(\partial/\partial x)$ can be derived in a standard way from the commutation relation (26) (see, e.g., [33, 34]). Instead of the standard approach, we will use results obtained above and calculate (v, v) in Eq. (14) for $\rho = |\psi|^2$ as follows:

$$\begin{aligned}(v, v) &= \int \frac{1}{|\psi|^2}\left(\frac{\partial\rho}{\partial x}\right)^2 dV \\ &= 4\int \frac{1}{|\psi|^2}\left[\mathrm{Re}\left(\psi^*\frac{\partial\psi}{\partial x}\right)\right]^2 dV = 4\int\left(\frac{\partial s_2}{\partial x}\right)^2 e^{-2s_2}dV.\end{aligned} \tag{27}$$

Using this result in Eq. (12) for $n = 0$, we get the uncertainty relation in the form

$$\int x^2|\psi|^2 dV \int \frac{1}{|\psi|^2}\left[\mathrm{Re}\left(\psi^*\frac{\partial\psi}{\partial x}\right)\right]^2 dV \ge \frac{1}{4}. \tag{28}$$

It follows from Eq. (27) that the last integral in Eq. (28) depends only on the function s_2, giving the form of the probability distribution ρ, and does not depend on s_1. Thus, in contrast to the usual uncertainty relations in quantum mechanics, uncertainty relation (28) does not depend on s_1.

Now we compare the last uncertainty relation with the usual uncertainty relations. First we see that

$$\begin{aligned}\int\left(\frac{\partial s_2}{\partial x}\right)^2 e^{-2s_2}dV &\le \int\left[\left(\frac{\partial s_1}{\partial x}\right)^2 + \left(\frac{\partial s_2}{\partial x}\right)^2\right]e^{-2s_2}dV \\ &= \int\left|-i\frac{\partial\psi}{\partial x}\right|^2 dV.\end{aligned} \tag{29}$$

Substituting the right-hand side of this inequality into Eq. (12) for $n = 0$, we get an uncertainty relation that looks more familiar than Eq. (28), i.e.,

$$\int x^2|\psi|^2 dV \int\left|-i\frac{\partial\psi}{\partial x}\right|^2 dV \ge \frac{1}{4}. \tag{30}$$

It is seen from Eq. (29) that this more usual form of the uncertainty relation depends both on s_1 and s_2, as well as that the left-hand side of Eq. (30) is greater than or equal to that of Eq. (28).

5. HEISENBERG UNCERTAINTY RELATION

The uncertainty relation (30) can be further generalized and even more general forms of the uncertainty relations can be obtained. Using the integration by parts and the condition $x|\psi|^2 \longrightarrow 0$ for $x \longrightarrow \pm\infty$, Eq. (24) can be rewritten as

$$\begin{aligned}&\int[(x-a)\psi]^*\left[\frac{\partial\psi}{\partial x} - ib\psi\right]dV \\ &+ \int\left[\frac{\partial\psi}{\partial x} - ib\psi\right]^*[(x-a)\psi]dV = -1,\end{aligned} \tag{31}$$

where a and b are arbitrary real constants. This equation can be further written as

$$(u, v) + (v, u) = -1, \tag{32}$$

where

$$u = (x-a)\psi \tag{33}$$

and

$$v = \partial\psi/\partial x - ib\psi. \tag{34}$$

Using the property $(v, u) = (u, v)^*$, we get

$$2\,\mathrm{Re}(u, v) = -1. \tag{35}$$

Calculating the square of the last equation, we get successively

$$\begin{aligned}1 = 4[\mathrm{Re}(u, v)]^2 &\le 4\{[\mathrm{Re}(u, v)]^2 + [\mathrm{Im}(u, v)]^2\} \\ &= 4|(u, v)|^2.\end{aligned} \tag{36}$$

Using this equation and the Schwarz inequality (11), we obtain the result

$$(u, u)(v, v) \ge 1/4 \tag{37}$$

which can be rewritten in form of the uncertainty relation

$$\int (x-a)^2|\psi|^2 dV \int \left|-i\frac{\partial\psi}{\partial x} - b\psi\right|^2 dV \geq \frac{1}{4}. \quad (38)$$

This general form of the uncertainty relation is valid for any real numbers a and b.

An interesting question is to find the values of a and b leading to the smallest value of the left-hand side of the last uncertainty relation. The minimum of the left-hand side of Eq. (38) is obtained for

$$a = \int \psi^* x \psi dV = \langle x \rangle \quad (39)$$

and

$$b = \int \psi^* \left(-i\frac{\partial\psi}{\partial x}\right) dV = \left\langle -i\frac{\partial}{\partial x}\right\rangle. \quad (40)$$

Except for the factor $\hbar^2$, the resulting uncertainty relation

$$\int (x-\langle x \rangle)^2|\psi|^2 dV \int \left|-i\frac{\partial\psi}{\partial x} - \left\langle -i\frac{\partial}{\partial x}\right\rangle\psi\right|^2 dV \geq \frac{1}{4} \quad (41)$$

agrees with the well-known Heisenberg uncertainty relation [33–35]

$$\int (x-\langle x \rangle)^2|\psi|^2 dV \int \left|-i\hbar\frac{\partial\psi}{\partial x} - \left\langle -i\hbar\frac{\partial}{\partial x}\right\rangle\psi\right|^2 dV \geq \frac{\hbar^2}{4}. \quad (42)$$

Therefore, the Heisenberg uncertainty relation corresponds to the smallest value of the left-hand side of a more general uncertainty relation (38), in which both the functions s_1 and s_2 are taken into account.

6. THIRD UNCERTAINTY RELATION

Now we want to clarify the question of whether it is possible to make the left-hand side of the uncertainty relation (41) even smaller. First, we see that the second integral in the last uncertainty relation is greater than or equal to the integral appearing in Eq. (27)

$$\begin{aligned} &\int \left|-i\frac{\partial\psi}{\partial x} - \left\langle -i\frac{\partial}{\partial x}\right\rangle\psi\right|^2 dV \\ &= \int \left[\left(\frac{\partial s_1}{\partial x}\right)^2 + \left(\frac{\partial s_2}{\partial x}\right)^2\right] e^{-2s_2} dV \\ &- \left(\int \frac{\partial s_1}{\partial x} e^{-2s_2} dV\right)^2 \geq \int \left(\frac{\partial s_2}{\partial x}\right)^2 e^{-2s_2} dV. \end{aligned} \quad (43)$$

Here, the equality is obtained only if the function s_1 does not depend on x. In deriving the preceding relation, we used the result

$$\begin{aligned} &\int \frac{\partial s_2}{\partial x} e^{-2s_2} dV = -\frac{1}{2}\int \frac{\partial e^{-2s_2}}{\partial x} dV \\ &= -\frac{1}{2}\int \frac{\partial |\psi|^2}{\partial x} dV = -\frac{1}{2}\int |\psi|^2\Big|_{x=-\infty}^{\infty} dy dz = 0. \end{aligned} \quad (44)$$

Further, we can repeat the procedure used in Section 2 with $u = (x - \langle x \rangle)\psi$ and obtain the uncertainty relation

$$\int (x-\langle x \rangle)^2|\psi|^2 dV \int \frac{1}{|\psi|^2}\left[\mathrm{Re}\left(\psi^*\frac{\partial\psi}{\partial x}\right)\right]^2 dV \geq \frac{1}{4}, \quad (45)$$

which is a generalization of the uncertainty relation (28). The left-hand side of this uncertainty relation is less than or equal to that of Eq. (41) and the equality is obtained for s_1 independent of x. Therefore, the Heisenberg uncertainty relation (42) can be replaced by the uncertainty relation

$$\begin{aligned} &\int (x-\langle x \rangle)^2|\psi|^2 dV \\ &\times \int \frac{1}{|\psi|^2}\left\{\mathrm{Re}\left[\psi^*\left(-i\hbar\frac{\partial\psi}{\partial x}\right)\right]\right\}^2 dV \geq \frac{\hbar^2}{4}. \end{aligned} \quad (46)$$

If the function s_1 depends on x, the left-hand side of this relation can have a smaller value than that in the Heisenberg uncertainty relation (42).

Similar to the commutation relations, we note that these uncertainty relations are a direct consequence of the integration by parts applied to the definition of the mean value (3) and the Born rule (17). It is also seen that similar uncertainty relations can be obtained not only in quantum mechanics, but in any probabilistic theory of a similar form. If the probabilistic character of measurements can be neglected (as for example in case of classical mechanics), the uncertainty relations disappear analogously to the commutation relations. Another general discussion of the uncertainty relations can be found, for example, in [36–38].

There are two important operators appearing in the commutation and uncertainty relations discussed above, i.e., the coordinate x and the operator $-i(\partial/\partial x)$. Except for $\hbar$, the operator $-i(\partial/\partial x)$ equals the momentum operator $\hat{p}_x = -i\hbar(\partial/\partial x)$ known from quantum mechanics. The momentum operator will be discussed in more detail in Section 9.

7. VECTOR POTENTIAL

It is worth noting that Eq. (31) also remains valid in the case that the constant b is replaced by a real function $b = f_x(\mathbf{r}, t)$. This means that the operator $-i(\partial\psi/\partial x)$ can be replaced by the operator $-i(\partial\psi/\partial x) - f_x$ and the commutation relation (26) and the uncertainty relation (38) can be further generalized. Therefore, general structure of the theory remains preserved for any real function f_x. It is the mathematical result only. In physics, the functions f_x, f_y, and f_z can, for example, correspond to the components of the electromagnetic vector potential $\mathbf{A} = (A_x, A_y, A_z)$ multiplied by the charge q of the particle. Except for $\hbar$, it agrees with the rule $-i\hbar\nabla \longrightarrow -i\hbar\nabla - q\mathbf{A}$ for including the vector potential $\mathbf{A}$ into quantum theory (for charge, see the end of Section 10).

It is seen that possibility to include the vector potential into the theory is another general consequence of the definition (3), integration by parts, and the Born rule (17).

8. PROBABILITY DENSITY CURRENT

To describe physical systems, we have to specify not only the form of the probability distribution $\rho = \rho(\mathbf{r}, t)$ determining the mean values $\langle x^n \rangle$ at time t, but also the motion of the measured system in space. Information about the motion of the system in space can be described by means of the function s_1 in Eq. (18), which could be an arbitrary real function until now. Here, we can proceed similarly as in continuum mechanics, where not only the density ρ of the continuum, but also the corresponding density current $\mathbf{j} = \rho\mathbf{v}$, where $\mathbf{v}$ is the velocity of the continuum, are used to describe the state of the system.

Since we use the probability density ρ here, the corresponding current will have meaning of the probability density current. Analogously to continuum mechanics, we can define the probability density current j_k by the equation

$$j_k = \rho v_k, \quad k = 1, 2, 3, \tag{47}$$

where the vector v_k does not give the real velocity of the particle and has only a probabilistic meaning. Further, we can write the vector v_k by means of the gradient of the function s_1

$$v_k = (\hbar/m_0)(\partial s_1/\partial x_k), \tag{48}$$

where m_0 is the rest mass of the particle and $\hbar$ is a constant that determines the units used in measurement (see also [39]). The numerical value of $\hbar$ has to be found experimentally.

Equation (48) can be viewed as a probabilistic generalization of the classical expression

$$\mathbf{v} = \mathbf{p}/m_0 = (\nabla S)/m_0,$$

where $\mathbf{p}$ is the classical momentum and S is the classical action. One can also be inspired by the eikonal theory or take Eq. (48) as a purely mathematical trick leading to a simple mathematical expression between j_k and s_1. The spatial derivative in Eq. (48) could also be replaced by a different functional relation between j_k and s_1. However, using the spatial derivative in Eq. (48) has two important advantages in that it leads to the linearity of the expression for j_k in terms of ψ discussed below and contains the spatial derivatives $(\partial/\partial x_k)$ that already appeared in the commutation and uncertainty relations discussed in the preceding sections. Thus, formulation based on Eqs. (47) and (48) has important mathematical and physical advantages and ψ carries information on the motion of the measured system in space. We also note that the number of the quantities ρ and j_k, $k = 1, 2, 3$ equals the number of the space–time dimensions.

Using Eqs. (47), (48), we get successively

$$\begin{aligned} j_k &= \rho v_k = \rho\frac{\hbar}{m_0}\frac{\partial s_1}{\partial x_k} = \sqrt{\rho}e^{-is_1}\sqrt{\rho}e^{is_1}\frac{\hbar}{m_0}\frac{\partial s_1}{\partial x_k} \\ &= \frac{\hbar}{m_0}\left[\sqrt{\rho}e^{-is_1}(-i)\frac{\partial(\sqrt{\rho}e^{is_1})}{\partial x_k} + \frac{i}{2}\frac{\partial\rho}{\partial x_k}\right]. \end{aligned} \tag{49}$$

Now, we can use the complex probability amplitude (18) and get

$$j_k = \frac{\hbar}{m_0}\left[\psi^*\left(-i\frac{\partial\psi}{\partial x_k}\right) + \frac{i}{2}\frac{\partial\rho}{\partial x_k}\right]. \tag{50}$$

However, the probability density current is real. Therefore, by calculating the real part of the last expression, we obtain the final formula for j_k as

$$\begin{aligned} j_k &= \frac{\hbar}{2m_0}\left[\psi^*\left(-i\frac{\partial\psi}{\partial x_k}\right) + \text{c.c.}\right] \\ &= \frac{\hbar}{2m_0 i}\left(\psi^*\frac{\partial\psi}{\partial x_k} - \psi\frac{\partial\psi^*}{\partial x_k}\right). \end{aligned} \tag{51}$$

This formula agrees with the expression for the probability density current known from quantum mechanics [33, 34].

The manner of deriving Eq. (51) shows that the relation between the probability amplitude and probability density in form of the Born rule (18) yields not only physically meaningful quantities in the uncertainty relations, but also leads to the simple expression for the probability density current (51).

It is seen from Eq. (51) that to obtain nonzero j_k, the probability amplitude ψ must be complex. In agreement with the rules known from quantum mechanics, the probability amplitudes ψ and $\psi\exp(i\alpha)$, where α is a real constant, yield the same probability density ρ and probability density current j_k.

It is to be noted that the expression for the probability amplitude in the form

$$\psi = e^{is} = e^{is_1 - s_2} \tag{52}$$

describes two different aspects of physical measurements. The first aspect, represented by the function s_2, is related to the probability density ρ via the equation $s_2 = -(1/2)\ln\rho$. The second one, represented by s_1, is related to the probability density current $\mathbf{j} = \rho\hbar\nabla s_1/m_0$. In other words, they are two different kinds of physical information carried by two different functions s_1 and s_2. In this sense, the probability amplitude $\psi = \exp(is_1 - s_2)$ represents the state of the system as it is known from quantum mechanics.

9. FISHER INFORMATION, KINETIC ENERGY, AND MOMENTUM OPERATOR

We note that the integral (v, v) in Eq. (14), which appears in the uncertainty relation (28)

$$\begin{aligned} I_x &= (v, v) = \int \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 dV \\ &= 4\int\left(\frac{\partial|\psi|}{\partial x}\right)^2 dV = 4\int \rho\left(\frac{\partial s_2}{\partial x}\right)^2 dV \end{aligned} \tag{53}$$

is known as the Fisher information, which characterizes the probability distribution ρ with respect to the variable x [15, 24, 40–42]. The following inequalities can be obtained from Eq. (12):

$$\langle x^2 \rangle \geq 1/I_x \tag{54}$$

and

$$\langle x^{2n+2} \rangle \geq [(n+1)\langle x^n \rangle]^2/I_x, \quad n = 1, 2, 3, \ldots. \tag{55}$$

Therefore, the Fisher information I_x has simple physical meaning. It determines the lower bounds to the quantities $\langle x^{2n+2} \rangle$, $n = 0, 1, 2, \ldots$, characterizing the results of measurements. The larger the value of I_x, the smaller values of $\langle x^{2n+2} \rangle$, $n = 0, 1, 2, \ldots$ are that can be obtained by measurement. In three dimensions, the corresponding three-dimensional Fisher information can be written as

$$I = I_x + I_y + I_z = 4\int \rho(\nabla s_2)^2 dV. \tag{56}$$

However, these Fisher informations depend on s_2 only and do not take into consideration the probability current j_k represented by s_1. Therefore, the Fisher informations I_x and I can, in physics, be generalized as [15]

$$\begin{aligned} I'_x &= 4\int \rho\left[\left(\frac{\partial s_1}{\partial x}\right)^2 + \left(\frac{\partial s_2}{\partial x}\right)^2\right] dV \\ &= 4\int\left|-i\frac{\partial \psi}{\partial x}\right|^2 dV \geq I_x \end{aligned} \tag{57}$$

and

$$\begin{aligned} I' &= I'_x + I'_y + I'_z = 4\int \rho[(\nabla s_1)^2 + (\nabla s_2)^2] dV \\ &= 4\int |-i\nabla\psi|^2 dV \geq I. \end{aligned} \tag{58}$$

The last integral also appears in the expression for kinetic energy known from quantum mechanics as follows:

$$T = \frac{\hbar^2}{2m_0}\int |-i\nabla\psi|^2 dV = \frac{\hbar^2 I'}{8m_0}. \tag{59}$$

Therefore, the kinetic energy T in quantum mechanics is proportional to the generalized Fisher information I', in which both gradients ∇s_1 (related to the probability density current j) and ∇s_2 (related to the probability density ρ) are taken into account.

In classical mechanics, the probability density $\rho(\mathbf{r}, t)$ is narrow enough that it can be replaced by the function $\delta(\mathbf{r} - \langle \mathbf{r} \rangle)$, where $\langle \mathbf{r} \rangle = \langle \mathbf{r} \rangle(t)$ is the classical trajectory. Then, integration in Eq. (59) disappears and one can assume that s_2 has maximum at the classical trajectory and $\nabla s_2(\langle \mathbf{r} \rangle) = 0$. In this case, the kinetic energy depends only on the function $S_1(\langle \mathbf{r} \rangle, t) = \hbar s_1(\langle \mathbf{r} \rangle, t)$ playing the role of the classical action S (see Section 13).

The uncertainty relation (38) can be for $a = \langle x \rangle$ and $b = 0$ written in the form

$$\langle (x - \langle x \rangle)^2 \rangle \geq 1/I'_x, \tag{60}$$

which is more general than Eq. (54). In classical mechanics, the kinetic energy related to s_1 and the Fisher information I' are very large and the mean square displacement $\langle (x - \langle x \rangle)^2 \rangle$ is small enough that the classical trajectories can be introduced.

At the end of this section, we will make a few remarks on the momentum operator. In contrast to standard quantum mechanics, the Hermitian operator $-i\hbar\nabla$ need not be postulated in our approach. Its appearance in the commutation relations, uncertainty relations, probability density current, and the generalized Fisher information indicates its important role in the theory. Further, relation (51) between the probability density current $\mathbf{j}$ and the operator $-i\hbar\nabla$ shows that this operator can be used for describing the motion of the measured system in space. It agrees with quantum mechanics where the operator $\hat{\mathbf{p}} = -i\hbar\nabla$ is postulated as the momentum operator. The concrete physical meaning of the operator $\hat{\mathbf{p}}$, the momentum operator, can be clarified in different ways. The best approach is probably based on the transition to classical mechanics. Performing this transition, the quantity $\int |-i\hbar\nabla\psi|^2 dV/(2m_0)$ appearing in Eq. (59) becomes the kinetic energy in the Hamilton–Jacobi equation $(\nabla S)^2/(2m_0)$, where S is the classical action (see Sections 12, 13). Taking into account that $\mathbf{p} = \nabla S$ is the classical momentum, the operator $\hat{\mathbf{p}} = -i\hbar\nabla$ can be denoted as the momentum operator.

10. PROBABILITY DENSITY AND TIME

Time can be discussed analogously to the spatial coordinates; however, there are some important differences that have to be respected.

Assuming that the initial conditions for ψ at time $t = t_0$ are given, the probability amplitude $\psi(\mathbf{r}, t)$, $t > t_0$ gives the probabilistic description of measurements made at later times. Therefore, time evolution has a unidirectional character from given initial conditions to the relative probability of the results of (yet unperformed) measurements at later times. If such a measurement is actually performed, this probabilistic description must be replaced by a concrete result obtained from the performed measurement. The basis of two different parts of the evolution scheme used in quantum mechanics are as follows:

(1) Evolution between the initial conditions at $t = t_0$ and time $t > t_0$ of the following measurement. This time evolution is described by the equations of motion.

(2) The reduction or collapse of the wave function at time t of the actually performed measurement.

In this paper, we are interested mainly in equations of motion, i.e., in the first part of this evolution scheme. A detailed microscopic description of the reduction of the probability amplitude is not needed here and can be found, for instance, in [27, 28] (see also [43]).

In the case of the spatial coordinates discussed in Section 2, we investigated the bound states obeying the normalization condition

$$\int |\psi(\mathbf{r}, t)|^2 dV = 1 \tag{61}$$

valid at all times; i.e., we assumed an infinite lifetime for the investigated system. We also assumed that the mean values

$$\langle x^n \rangle = \int x^n |\psi(\mathbf{r}, t)|^2 dV \tag{62}$$

are finite.

In the case of time, we would like to proceed similarly as for the spatial coordinates. We would like to introduce the mean values

$$\langle t^n \rangle = \int\int_{t_0}^{\infty} t^n |\psi(\mathbf{r}, t)|^2 dV dt, \tag{63}$$

where, in agreement with our understanding of the arrow of time, time integration is carried out for $t \geq t_0$. Then we want to derive the corresponding commutation and uncertainty relations for time and energy and discuss the scalar potential, antiparticles, and equations of motion (Sections 10–12).

However, taking the usual normalization condition (61), i.e., assuming an infinite lifetime, the mean values $\langle t^n \rangle$ tend to infinity and an approach similar to that in case of the spatial coordinates cannot be used.

It is obvious that the validity of the normalization condition (61) at all times is, from the physical point of view, a rather limiting assumption since it means that the investigated system cannot change its state (for example, to go from the excited state to the ground state). Most likely, the simplest solution of this problem is to assume that the lifetime of the investigated system is finite and replace the probability amplitude ψ by the function

$$\chi(\mathbf{r}, t) = \psi(\mathbf{r}, t)\eta(t). \tag{64}$$

Here, $\eta(t)$ is a real decaying function normalized by the condition

$$\int_{t_0}^{\infty} |\eta(t)|^2 dt = 1, \tag{65}$$

for which the integrals

$$\langle t^n \rangle = \int_{t_0}^{\infty} t^n |\eta(t)|^2 dt \tag{66}$$

have finite values and $\psi(\mathbf{r}, t)$ obey Eq. (61). As a concrete example of a function obeying these conditions, we can take the exponential

$$\eta(t) = e^{-(t-t_0)/(2\tau)}/\sqrt{\tau}, \quad \tau > 0, \tag{67}$$

which is often used in quantum mechanics for describing a finite lifetime.

Then, repeating essentially the same procedure as for the spatial coordinates, we can define the square of the mean lifetime $\langle t^2 \rangle$, get the operator $i(\partial/\partial t)$, obtain the corresponding time commutation and uncertainty relations, and introduce the scalar potential. After performing all calculations in the space–time region, the transition to standard quantum mechanics of particles having infinite lifetimes can be made by assuming that $\eta(t)$ changes very slowly in time or $\tau \longrightarrow \infty$. At the same time, the normalization condition over the space–time,

$$\int\int_{t_0}^{\infty} |\chi(\mathbf{r}, t)|^2 dV dt = 1 \tag{68}$$

has to be replaced by the standard normalization condition (61).

Because of the analogy of this approach to that for the spatial coordinates, we will present here only the most important steps of this discussion. By analogy with Eq. (52), we write χ in the form

$$\chi = e^{is_1 - s_2}. \tag{69}$$

Then, we define the time component of the probability density current by the equation analogous to Eqs. (47), (48) as

$$j_t = -\rho(\hbar/m_0)(\partial s_1/\partial t) \tag{70}$$

and obtain an expression similar to Eq. (51), i.e.,

$$j_t = -\frac{\hbar}{2m_0 i}\left[\chi^*\left(\frac{\partial \chi}{\partial t}\right) + \text{c.c.}\right]. \tag{71}$$

We note that the sign in the definition of j_t is opposite to that in case of j_k. It corresponds to different signs of the time and spatial parts of the metric in the special relativity. Equation (70) corresponds to the time component of the probability density current

$$j_0 = \mathrm{Re}[\psi^* i\hbar(\partial\psi/\partial x_0)]/m_0$$

known from relativistic quantum mechanics [33, 34], where $x_0 = ct$.

Further, by analogy with Eq. (25), we get the equation

$$\iint_{t_0}^{\infty}\left[\left(i\frac{\partial\chi}{\partial t}\right)^* t\chi - (t\chi)^*\left(i\frac{\partial\chi}{\partial t}\right)\right]dVdt = i. \quad (72)$$

Similarly to the constant b in Eq. (31), we can introduce also a real constant b' into this equation as follows:

$$\iint_{t_0}^{\infty}\left[\left(i\frac{\partial\chi}{\partial t} - b'\chi\right)^* t\chi - (t\chi)^*\left(i\frac{\partial\chi}{\partial t} - b'\chi\right)\right]dVdt = i. \quad (73)$$

In contrast to the spatial coordinate x, where we could introduce an arbitrary shift a (see Eq. (31)), the time integration in the last equations runs from time $t = t_0$, when the initial conditions were given, and cannot be arbitrarily shifted. The corresponding time uncertainty relation can be written in a form analogous to Eq. (38) as follows:

$$\iint_{t_0}^{\infty} t^2|\chi|^2 dVdt \iint_{t_0}^{\infty}\left|i\frac{\partial\chi}{\partial t} - b'\chi\right|^2 dVdt \ge \frac{1}{4}. \quad (74)$$

The minimum of the left-hand side of Eq. (74) is obtained for

$$b' = \frac{1}{2}\left[\iint_{t_0}^{\infty}\chi^* i\frac{\partial\chi}{\partial t}dVdt + \mathrm{c.c.}\right]. \quad (75)$$

We note that Eq. (74) is also valid if a real constant b' is replaced by a real function $f_0(\mathbf{r}, t)$.

For $b' = 0$, we obtain the uncertainty relation

$$\langle t^2\rangle \ge 1/I_t'' \quad (76)$$

between the mean square time

$$\langle t^2\rangle = \iint_{t_0}^{\infty} t^2|\chi|^2 dVdt \quad (77)$$

and the time Fisher information

$$I_t'' = 4\iint_{t_0}^{\infty}\left|i\frac{\partial\chi}{\partial t}\right|^2 dVdt, \quad (78)$$

where the symbol $''$ denotes integration over space–time. It is seen from Eq. (76) that the time Fisher information I_t'' gives the lower bound to the mean square time.

To illustrate the meaning of the uncertainty relation (74), we consider the decaying probability amplitude

$$\chi(\mathbf{r}, t) = (1/\sqrt{\tau})e^{[-i\omega t - (t - t_0)/(2\tau)]}\psi(\mathbf{r}), \quad t \ge t_0, \quad (79)$$

where the spatial part of the probability amplitude ψ is normalized by the usual condition

$$\int|\psi(\mathbf{r})|^2 dV = 1.$$

In this case, we obtain the following from Eqs. (74), (75):

$$\langle t^2\rangle = \iint_{t_0}^{\infty} t^2|\chi|^2 dVdt = 2\tau^2, \quad (80)$$

$b' = \omega$, and

$$\iint_{t_0}^{\infty}\left|i\frac{\partial\chi}{\partial t} - b'\chi\right|^2 dVdt = \frac{1}{4\tau^2}. \quad (81)$$

Therefore, inequality (74) yields the relation between the mean square time $\langle t^2\rangle = 2\tau^2$ and the square of the absolute value of the imaginary part of the complex frequency $\omega - i/2\tau$ and has the physical meaning of the well-known time–energy uncertainty relation (see, e.g., [33, 34, 44]).

Now, we will discuss different roles of time and spatial coordinates in quantum mechanics (see also [45]). In quantum mechanics, it is assumed that the system is prepared in the state given by the initial conditions for the wave function $\psi(\mathbf{r}, t)$ in the whole space at time $t = t_0$. Then, equations of motion are used to calculate the wave function at later times and to determine the relative probability of results of future measurements made on the system. For this reason, the spatial coordinates and time have different roles in quantum mechanics and it is not surprising that the coordinate–momentum and time–energy uncertainty relations have different characters. The former relation gives one the relation between the mean square deviations of the coordinate and the momentum from their mean values and can be used for the infinite, as well as the finite, lifetime. The latter uncertainty relation is the relation between the mean lifetime (no mean square deviation) and the imaginary part of the energy (i.e., the width of the corresponding energy level) and can be used for system having a finite lifetime only.

11. POTENTIALS AND ANTIPARTICLES

In agreement with our understanding of the arrow of time from $t = t_0$ to $t > t_0$, we assume that the direct physical meanings only have the probability amplitudes corresponding to the non-negative values of the time component of the probability density current integrated over the whole space as follows:

$$\int j_t dV = -\frac{\hbar}{m_0}\int \rho \frac{\partial s_1}{\partial t} dV \geq 0. \tag{82}$$

If this quantity is negative, its sign can be reversed by the complex conjugation $\chi \longrightarrow \chi^*$, changing the sign of the phase s_1 and the probability density currents j_k and j_t. Performing this transformation, we get from Eq. (73) for $b' = f_0(\mathbf{r}, t)$,

$$\int\int_{t_0}^{\infty}\left[\left(i\frac{\partial \chi}{\partial t} + f_0\chi\right)^* t\chi - (t\psi)^*\left(i\frac{\partial \chi}{\partial t} + f_0\chi\right)\right] dVdt = i, \tag{83}$$

upon which we see that it leads to the change of the sign of the function f_0.

Analogous discussion can be also done for the spatial coordinates. As a result, the complex conjugation $\chi \longrightarrow \chi^*$ or $\psi \longrightarrow \psi^*$ leads to the change of sign of the functions f_0 and f_k, $k = 1, 2, 3$, which can be respected by putting $f_0 = qU$ and $f_k = qA_k$, where U and A_k can, for example, be the scalar and vector electromagnetic potentials and q denotes the charge of the particle. Therefore, the probability amplitudes ψ and ψ^* describe particles that differ by the sign of their charge and general structure of our probabilistic description and the unidirectional character of time contribute to understanding the existence of particles and antiparticles.

Except for $\hbar$, these conclusions agree with the well-known rules $i\hbar(\partial/\partial t) \longrightarrow i\hbar(\partial/\partial t) - qU$ and $-i\hbar\nabla \longrightarrow -i\hbar\nabla - q\mathbf{A}$ for including the electromagnetic potentials into quantum theory. These potentials representing different physical scenarios do not appear among the variables of the probability amplitude and describe non-quantized classical fields.

We have seen that to obtain results of Sections 2–11, no equations of motion have been needed. Thus, this part of the mathematical formalism of quantum mechanics follows directly from the probabilistic description of results of measurements. The Planck constant gives the units used in measurements and scales at which the probabilistic character of measurements is important. It must be determined experimentally.

12. EQUATIONS OF MOTION

First, we note that the physical content of the Fisher information—the characterization of the smoothness of the probability distribution—is similar to that of Shannon entropy. However, in contrast to the Shannon entropy, the Fisher information depends on the local properties of the probability distribution and can be used for deriving equations of motion [15].

To find equations of motion, we will require relativistic invariance of the theory. In this respect, our approach is different from that based on the principle of extreme physical information [15] or a minimum of the Fisher information [39]. Except for the end of this section, we will discuss free particles with no external fields.

To find relativistically invariant formulation, we use four time and spatial Fisher informations as follows:

$$I_t'' = 4\int\int_{t_0}^{\infty}\left|i\frac{\partial \chi}{\partial t}\right|^2 dVdt, \tag{84}$$

$$I_x'' = 4\int\int_{t_0}^{\infty}\left|-i\frac{\partial \chi}{\partial x}\right|^2 dVdt,$$

$$I_y'' = 4\int\int_{t_0}^{\infty}\left|-i\frac{\partial \chi}{\partial y}\right|^2 dVdt, \quad I_z'' = 4\int\int_{t_0}^{\infty}\left|-i\frac{\partial \chi}{\partial z}\right|^2 dVdt, \tag{85}$$

giving the lower bounds to $\langle t^2\rangle$, $\langle(x - \langle x\rangle)^2\rangle$, $\langle(y - \langle y\rangle)^2\rangle$, and $\langle(z - \langle z\rangle)^2\rangle$ (see Eqs. (60) and (76)). In contrast to the integration over all times used in [15, 39], time integration is performed here over the physically relevant region from the initial conditions at $t = t_0$ to infinity.

Due to the possibility of taking arbitrary physically reasonable initial conditions for $\chi(\mathbf{r}, t)$ at $t = t_0$, the spatial Fisher informations can have arbitrary values in the region

$$I_x'' \geq 0, \quad I_y'' \geq 0, \quad I_z'' \geq 0. \tag{86}$$

In this sense, the spatial Fisher informations are independent quantities. The time Fisher information is also non-negative, $I_t'' \geq 0$; however, its value is given not only by the initial conditions for $\chi(\mathbf{r}, t)$ at $t = t_0$, but also by the requirement of the relativistic invariance of the combined Fisher informations discussed below. From this point of view, the time Fisher information I_t'' is not an independent quantity.

To create the relativistic invariant from the Fisher informations, we will first take their linear combination in general form

$$I_t''/c^2 \pm (I_x'' + I_y'' + I_z'') = \text{const} \tag{87}$$

and assume that it has a value independent of the inertial system in which the measurement is performed. Here, c is the speed of light and we assume first that the plus or minus sign can be taken in this equation.

We note that for a very flat distribution ρ and the probability density current $j_k = 0$, the spatial Fisher informations approach zero and the time Fisher information equals $I_t'' = c^2$const (in physics, we usually say that the free particle is in its rest frame). It follows from here and the condition $I_t'' \geq 0$ that the constant const must be greater than or equal to zero

$$\text{const} \geq 0. \tag{88}$$

Further, we note that going from the inertial coordinate system in which $j_k = 0$ to another one where $j_k \neq 0$ (the particle moves with respect to this system), the sum of the spatial Fisher informations $I_x'' + I_y'' + I_z''$ (and the kinetic energy of the particle) increases. Assuming first the positive sign in front of the sum of the spatial Fisher informations, we see that for $I_x'' + I_y'' + I_z'' >$ const, the time Fisher information I_t'' would become negative in contradiction with its property $I_t'' \geq 0$. Therefore, to guarantee that the left-hand side of Eq. (87) equals const ≥ 0 and $I_t'' \geq 0$ for all values of $I_x'' + I_y'' + I_z'' \geq 0$, we have to take the time and spatial Fisher informations with a different sign and assume

$$I_t''/c^2 - (I_x'' + I_y'' + I_z'') = \text{const}. \quad (89)$$

Thus, our analysis based on the properties of the Fisher informations confirms that the signs of the metric used in Eq. (89) and the special theory of relativity must be different for time and the spatial coordinates. We note that in contrast to the constant const, the Fisher informations I_t'', I_x'', I_y'', and I_z'' can have different values in different inertial systems.

Rewriting Eq. (89) in the form

$$\int_{t_0}^{\infty}\!\!\int \left| i\hbar \frac{\partial \chi}{\partial t} \right|^2 dV dt = c^2 \int_{t_0}^{\infty}\!\!\int |-i\hbar \nabla \chi|^2 dV dt + \frac{\text{const}\,\hbar^2 c^2}{4}, \quad (90)$$

it can be compared with the well-known Einstein equation

$$E^2 = c^2 p^2 + m_0^2 c^4. \quad (91)$$

We will consider a free particle described by the probability amplitude

$$\chi(\mathbf{r}, t) = e^{(Et - \mathbf{pr})/i\hbar} [e^{-(t - t_0)/2\tau}/\sqrt{\tau}] \left(\frac{\alpha}{\pi}\right)^{3/4} e^{-\alpha r^2/2}, \quad \tau > 0, \quad \alpha > 0. \quad (92)$$

Here, the first exponential represents the probability amplitude describing a free particle with the same probability density to find the particle anywhere in time and space, E and $\mathbf{p}$ are a real number and a real vector. To obey the normalization condition (68), we multiplied the first exponential by two additional ones corresponding to a very long lifetime τ and a very small spatial damping factor α guaranteeing that the integrals in Eq. (90) are finite. Calculating all integrals in Eq. (90) and putting $\tau \longrightarrow \infty$ and $\alpha \longrightarrow 0$ at the end of the calculation, we get

$$E^2 = c^2 p^2 + \text{const}\,\hbar^2 c^2/4. \quad (93)$$

It is seen that this equation agrees with Eq. (91) for

$$\text{const} = 4 m_0^2 c^2/\hbar^2. \quad (94)$$

This discussion shows again that the operators $i\hbar(\partial/\partial t)$ and $-i\hbar\nabla$ in Eq. (90) can be denoted as the energy and momentum operators, respectively.

In the rest of this section, we will derive the most important equations of motion of quantum mechanics in a similar way as in [15]. Using the normalization condition (68), we rewrite Eq. (89) in the form

$$J[\chi] = \int_{t_0}^{\infty}\!\!\int \left(\frac{1}{c^2} \left| \frac{\partial \chi}{\partial t} \right|^2 - |\nabla \chi|^2 - \frac{\text{const}}{4} |\chi|^2 \right) dV dt = 0. \quad (95)$$

Equation of motion can be found from the condition that $J[\chi]$ is extremal with respect to χ,

$$\delta J[\chi] = 0 \quad (96)$$

or

$$\int_{t_0}^{\infty}\!\!\int \left(\frac{1}{c^2} \frac{\partial \delta\chi^*}{\partial t} \frac{\partial \chi}{\partial t} - \nabla \delta\chi^* \nabla \chi - \frac{\text{const}}{4} \delta\chi^* \chi + \text{c.c.} \right) dV dt = 0, \quad (97)$$

where δ denotes the variation of the corresponding quantity.

Now, we perform integration by parts with respect to time in the first term and with respect to the spatial coordinates in the second one and assume that the variations of χ equal zero at the borders of the integration region

$$\delta\chi|_{t = t_0}^{t = \infty} = 0, \quad (98)$$

$$\delta\chi|_{x_k = -\infty}^{x_k = \infty} = 0, \quad k = 1, 2, 3. \quad (99)$$

Then we get the equation

$$\int_{t_0}^{\infty}\!\!\int \delta\chi^* \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\text{const}}{4} \right) \chi dV dt + \text{c.c.} = 0, \quad (100)$$

which has to be obeyed for arbitrary variations $\delta\chi$ and $\delta\chi^*$. This yields the equation for the probability amplitude χ in the form

$$[\Delta - (1/c^2)(\partial^2/\partial t^2) - \text{const}/4]\chi = 0 \quad (101)$$

and its complex conjugate. Using Eq. (64), the last equation becomes

$$(\Delta\psi)\eta - \frac{1}{c^2}\left(\frac{\partial^2 \psi}{\partial t^2} \eta + 2 \frac{\partial \psi}{\partial t} \frac{d\eta}{dt} + \psi \frac{d^2 \eta}{dt^2} \right) - \frac{\text{const}}{4} \psi\eta = 0. \quad (102)$$

For a particle with an infinite lifetime, we assume that the time derivatives of the function η go to zero, $d\eta/dt \longrightarrow 0$ and $d^2\eta/dt^2 \longrightarrow 0$. Then, using the probability amplitude (64) and the constant const in form (94) we obtain the usual Klein–Gordon equation for a free particle

$$[\Delta - (1/c^2)(\partial^2/\partial t^2) - m_0^2c^2/\hbar^2]\psi = 0. \tag{103}$$

At the same time, the normalization condition (68) has to be replaced by Eq. (61).

The nonrelativistic time Schrödinger equation for a free particle

$$i\hbar\partial\varphi/\partial t = (-\hbar^2/2m_0)\Delta\varphi \tag{104}$$

can be obtained from the Klein–Gordon equation (103) by using the transformation

$$\psi = e^{m_0c^2t/i\hbar}\varphi, \tag{105}$$

where φ is the probability amplitude appearing in the Schrödinger equation. This transition is well known and will not be discussed here (see, e.g., [33, 34]). For different derivation of the Klein–Gordon equation and time Schrödinger equation, see [39].

The Dirac equation can be derived by taking the probability amplitude χ in Eq. (95) in form of a column vector with four components

$$\iint_{t_0}^{\infty}\left(\frac{1}{c^2}\frac{\partial\chi^+}{\partial t}\frac{\partial\chi}{\partial t} - \sum_{k=1}^{3}\frac{\partial\chi^+}{\partial x_k}\frac{\partial\chi}{\partial x_k} - \frac{\text{const}}{4}\chi^+\chi\right)dVdt = 0, \tag{106}$$

where the cross denotes the Hermitian conjugate. Inserting the γ^μ matrices with the well-known properties [33] into this equation and using Eq. (94), we get the integral

$$K[\chi] = \iint_{t_0}^{\infty}\left[\frac{1}{c^2}\left(\gamma^0\frac{\partial\chi}{\partial t}\right)^+\left(\gamma^0\frac{\partial\chi}{\partial t}\right) - \sum_{k=1}^{3}\left(\gamma^k\frac{\partial\chi}{\partial x_k}\right)^+\left(\gamma^k\frac{\partial\chi}{\partial x_k}\right) - \frac{m_0^2c^2}{\hbar^2}\chi^+\chi\right]dVdt = 0 \tag{107}$$

analogous to the integral $J[\chi]$. Then, using properties of the γ^μ matrices, performing the integration by parts and using boundary conditions analogous to Eqs. (98) and (99) we get (see also [15])

$$K[\chi] = \iint_{t_0}^{\infty}\left(\frac{\gamma^0}{c}\frac{\partial\chi}{\partial t} - \sum_{k=1}^{3}\gamma^k\frac{\partial\chi}{\partial x_k} - i\frac{m_0c}{\hbar}\chi\right)^+ \times\left(\frac{\gamma^0}{c}\frac{\partial\chi}{\partial t} + \sum_{k=1}^{3}\gamma^k\frac{\partial\chi}{\partial x_k} + i\frac{m_0c}{\hbar}\chi\right)dVdt = 0. \tag{108}$$

The operator in the first set parentheses is the hermitian conjugate of that in the second set. Equation (108) can be obeyed by assuming that the expression in the first or second set of parentheses equals zero. After the substitution $\chi = \psi\eta$ and assumption $d\eta/dt \longrightarrow 0$, the latter condition yields the Dirac equation in the form

$$\frac{\gamma^0}{c}\frac{\partial\psi}{\partial t} + \sum_{k=1}^{3}\gamma^k\frac{\partial\psi}{\partial x_k} + \frac{im_0c}{\hbar}\psi = 0. \tag{109}$$

We have seen that the requirement of the relativistic invariance of the linear combination of the time and spatial Fisher informations yields the basic equations of motion of quantum mechanics. The scalar and vector potentials U and $\mathbf{A}$ can be included into the equations of motion by means of the usual rules

$$i\hbar(\partial/\partial t) \longrightarrow i\hbar(\partial/\partial t) - qU$$

and

$$-i\hbar\nabla \longrightarrow -i\hbar\nabla - q\mathbf{A}$$

discussed above.

13. CLASSICAL MECHANICS

To derive the Hamilton–Jacobi equation of classical mechanics, we proceed as follows. The probability amplitude φ appearing in the Schrödinger equation can be taken in a form analogous to Eq. (52),

$$\varphi(\mathbf{r}, t) = e^{i(S_1 + iS_2)/\hbar} = e^{iS_1/\hbar}e^{-S_2/\hbar}, \tag{110}$$

where the phase of the probability amplitude is now expressed in the units $\hbar$ and the probability amplitude is normalized by means of the usual condition $\int|\varphi|^2dV = 1$. The time Schrödinger equation with the potential energy qU equals

$$i\hbar\partial\varphi/\partial t = (-\hbar^2/2m_0)\Delta\varphi + qU\varphi. \tag{111}$$

Multiplying this equation by φ^* and integrating over the whole space, we get, after simple integration by parts,

$$i\hbar\int\varphi^*\frac{\partial\varphi}{\partial t}dV = \frac{1}{2m_0}\int|-i\hbar\nabla\varphi|^2dV + \int qU|\varphi|^2dV. \tag{112}$$

Substituting Eq. (110) into the last equation yields

$$\int \frac{\partial S_1}{\partial t} e^{-2S_2/\hbar} dV + i \int \frac{\partial S_2}{\partial t} e^{-2S_2/\hbar} dV + \frac{1}{2m_0} \int (\nabla S_1)^2 e^{-2S_2/\hbar} dV + \frac{1}{2m_0} \int (\nabla S_2)^2 e^{-2S_2/\hbar} dV + \int qU e^{-2S_2/\hbar} dV = 0. \tag{113}$$

Due to the normalization condition

$$\int |\varphi|^2 dV = \int \exp(-2S_2/\hbar) dV = 1,$$

the second integral in this equation equals zero and the imaginary term disappears from this equation.

Now we assume that the probability density

$$|\varphi|^2 = e^{-2S_2/\hbar} \tag{114}$$

has very small values everywhere except for in close vicinity to the point $\langle \mathbf{r} \rangle = \langle \mathbf{r} \rangle(t)$, where it achieves its maximum and the first derivatives of S_2 at this point equal zero

$$\nabla S_2 |_{\mathbf{r} = \langle \mathbf{r} \rangle} = 0. \tag{115}$$

In such a case, the probability density can be replaced by the δ-function

$$|\varphi|^2 = \delta(\mathbf{r} - \langle \mathbf{r} \rangle) \tag{116}$$

and the probabilistic character of the theory disappears. Therefore, the function S_2 describing the form of the probability distribution ρ does not appear in classical mechanics.

Then, straightforward use of Eqs. (113)–(116) leads to the Hamilton–Jacobi equation for the function S_1 in the variable $\langle \mathbf{r} \rangle$

$$\frac{\partial S_1(\langle \mathbf{r} \rangle, t)}{\partial t} + \frac{(\nabla S_1(\langle \mathbf{r} \rangle, t))^2}{2m_0} + qU(\langle \mathbf{r} \rangle, t) = 0. \tag{117}$$

In classical mechanics, the variable $\langle \mathbf{r} \rangle$ is usually replaced by the classical coordinate $\mathbf{r}$ and the function S_1 is called the action S

$$\frac{\partial S(\mathbf{r}, t)}{\partial t} + \frac{(\nabla S(\mathbf{r}, t))^2}{2m_0} + qU(\mathbf{r}, t) = 0. \tag{118}$$

We note that Eq. (116) corresponds to the limit $S_2 \gg \hbar$ or $\hbar \longrightarrow 0+$ in Eq. (114). Therefore, the function S_1 in Eq. (117) is in fact the first term of the expansion of S_1 into the power series in $\hbar$

$$S_1 = S_1|_{\hbar = 0} + \dots. \tag{119}$$

In this limit, the commutation and uncertainty relations disappear from the theory.

We have seen that the Hamilton–Jacobi equation follows from the probabilistic description of results of measurements in the limit of the δ-like probability densities ρ and the nonrelativistic approximation. As usual, the vector potential $\mathbf{A}$ can be included into the theory by means of the rule $\nabla S \longrightarrow \nabla S - q\mathbf{A}$ following from the rule $-i\hbar\nabla \longrightarrow -i\hbar\nabla - q\mathbf{A}$ discussed in Sections 7 and 11. The Hamilton–Jacobi equation can be also the starting point for obtaining the Hamilton variational principle of classical mechanics.

14. MANY-PARTICLE SYSTEMS

The starting point of discussion of the N particle system is a definition analogous to Eq. (3) in that

$$\langle \mathbf{r}_j \rangle = \int \mathbf{r}_j \rho(\mathbf{r}_1, \dots, \mathbf{r}_N, t) dV_1 \dots dV_N, \quad j = 1, \dots, N, \tag{120}$$

where ρ is the many-particle probability density and $\mathbf{r}_j$ are the coordinates of the jth particle. Then, discussion can be performed analogously to that given above and the probability amplitude, uncertainty and commutation relations, momentum operators, and density currents for all particles can be introduced. The scalar and vector potentials $U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ and $\mathbf{A}(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$, and antiparticles can also be discussed.

The Schrödinger equation for N free particles can be found from the many-particle generalization of the relativistic invariant (95)

$$\int_{t_0}^{\infty} \int \left(\frac{1}{c^2} \left| \frac{\partial \chi}{\partial t} \right|^2 - \sum_{j=1}^{N} |\nabla_j \chi|^2 - \sum_{j=1}^{N} \frac{m_j^2 c^2}{\hbar^2} |\chi|^2 \right) dV_1 \dots dV_N dt = 0, \tag{121}$$

where $\chi(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ is the N particle probability amplitude, and m_j denotes the rest mass of the jth particle. Using a similar approach to that in Section 12, we can then obtain the Schrödinger equation for N free particles as

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_{j=1}^{N} \frac{\hbar^2}{2m_j} \Delta_j \psi \tag{122}$$

and the Hamilton–Jacobi equation as

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{N} \frac{(\nabla_j S)^2}{2m_j} = 0. \tag{123}$$

It is seen that the probabilistic description of results of measurement and its relativistic invariance yield also the basic mathematical structure of the many-particle quantum mechanics.

15. POSTULATES OF QUANTUM MECHANICS

In this section, we will show that the basic formulation of the postulates of quantum mechanics can be obtained from the discussion given in the preceding sections.

It follows from the above discussion that the probability density $\rho(\mathbf{r}, t)$ and the probability density current $j_k(\mathbf{r}, t)$ can, in a mathematically simple and straightforward way, be represented by the probability amplitude $\psi(\mathbf{r}, t)$. Using the probability amplitude $\psi(\mathbf{r}, t)$, it is possible to calculate the mean values of the operators representing the coordinates x_k and momentum p_k, $k = 1, 2, 3$, and further physical quantities that can be expressed in terms of the coordinates and momentum (the kinetic energy, total energy, etc.). In this sense, the probability amplitude $\psi(\mathbf{r}, t)$ represents the state of the system. It agrees with the usual first postulate of quantum mechanics formulated in the coordinate representation.

According to some interpretations of quantum mechanics, the probability amplitude ψ only represents the subjective knowledge of the observer. However, since ψ carries the information on ρ and j_k, it contains the objective physical information on the system. A certain analogy with this situation can be found in statistical physics, where the statistical distribution functions describe physical reality in a different probabilistic sense. For this reason, we do not agree with such purely subjective interpretations.

Mean values of the coordinates and momentum can be calculated from the formulas (compare with Eqs. (39) and (40))

$$\langle x_k \rangle = \int \psi^* \hat{x}_k \psi dV \tag{124}$$

and

$$\langle p_k \rangle = \int \psi^* \hat{p}_k \psi dV. \tag{125}$$

Here, the coordinate and momentum operators equal $\hat{x}_k = x_k$ and $\hat{p}_k = -i\hbar(\partial/\partial x_k)$, where $\hbar$ depends on the units used in measurement. To obtain the Hamilton operator $\hat{H} = \hat{T} + \hat{V}$ appearing in the Schrödinger equation, the operators $\hat{x}_k$ and $\hat{p}_k$ can be formally substituted for the classical quantities x_k and p_k in the classical Hamilton function

$$\begin{aligned} H = T + V &= \frac{1}{2m_0}\sum_{k=1}^{3}\left(\frac{\partial S}{\partial x_k}\right)^2 + V \\ &= \frac{1}{2m_0}\sum_{k=1}^{3}(p_k)^2 + V, \end{aligned} \tag{126}$$

appearing in the Hamilton–Jacobi equation (118). It is obvious that a similar rule also applies for other physical quantities depending on x_k and p_k. This rule agrees with the second postulate of quantum mechanics.

For the sake of simplicity, the following discussion will be made in one dimension only. In our original x-representation, we assumed that we use an experimental apparatus that can measure the coordinate and the probability density of getting the coordinate x in measurement equals $\rho(x) = |\psi(x)|^2$ (see Eq. (3)). Let us assume now that we perform measurement with another experimental apparatus that measures the momentum and the probability of getting the value p of the momentum equals $|\varphi(p)|^2$. We also assume that the corresponding probability amplitudes ψ and φ are related by the unitary transformation U, preserving the norm of the probability distributions as follows:

$$\psi = U\varphi. \tag{127}$$

Then, the mean value of the coordinate x equals

$$\langle x \rangle = \int \psi^* x \psi dx = \int \varphi^*(U^{-1}xU)\varphi dp, \tag{128}$$

where the integration in the second integral is performed over p. This means that the coordinate operator in this momentum or p-representation is not diagonal, $\hat{x} = U^{-1}xU$. Assuming that the system that will be measured, it is prepared in the state described by the probability amplitude φ, the probability of getting the value x in the following measurement of the x coordinate equals

$$\langle x \rangle = \int x|\psi|^2 dx, \tag{129}$$

where

$$|\psi|^2 = |U\varphi|^2. \tag{130}$$

This result agrees with the third postulate of quantum mechanics, according to which the probability of getting the x value during the measurement of the coordinate on the system in the state described by the wave function φ can be written as $|\langle x|\varphi\rangle|^2$.

We note that the values of the x coordinate appearing in the definition of the mean value $\langle x \rangle$ or the values of the momentum p in measurement of the momentum are real quantities and the corresponding operators $\hat{x}$ or $\hat{p}$ are diagonal in their own x- or p-representation. After transition to another representation by means of the unitary representation, these operators are not diagonal; however, they remain hermitian. This conclusion also agrees with the rules of quantum mechanics.

After the measurement of the coordinate x is performed, yielding one concrete value x', the state of the system can be described by the probability amplitude $\delta(x - x')$ (the eigenfunction of the operator $\hat{x} = x$). This probability amplitude replaces the original one describing the state of the system immediately before the measurement and its time evolution until the time of the following measurement is given by the corresponding equation of motion. This change of the probability amplitude resulting from the performed measurement is known as the reduction or collapse of the wave function.

Requirement of the relativistic invariance of the generalized space–time Fisher information yields the usual fourth postulate of quantum mechanics postulating the equations of motion (see Section 12).

We have seen that the rules discussed above can be obtained from the probabilistic description of measurement of the coordinates and time and the relativistic invariance of the theory. More general formulation of the postulates of quantum mechanics can be obtained by generalizing the above discussion to further physical quantities.

16. CONCLUSIONS

In [13], Landé tried to at least partially illuminate the old problem of Einstein of why the world is a quantum world [46]. Landé started his discussion of the quantum world with three postulates, including (a) the symmetry of the probabilities of the transition from the state α to the state β, $P_{\alpha\beta} = P_{\beta\alpha}$; (b) correspondence between the actual microscopic and the ordinary probability law; (c) the covariance of dynamics requiring that only differences of physical quantities that are like the energy appear in observable quantities. Replacing the standard postulates by the postulates (a)–(c), he was able to obtain the momentum operator and commutation relations. Using the coordinate and the momentum operator, the Schrödinger equation can be obtained in the usual way from the classical mechanics.

In this paper, we have made an attempt to avoid this or any other set of similar postulates and clarify the basic structure of quantum mechanics by using the probabilistic description of measurements.

We started by assuming that the probabilistic results of measurement of the space coordinate x are given by the probability density $\rho(\mathbf{r}, t)$. Then, using integration by parts and the Schwarz inequality, we derived relations (12) containing an arbitrary complex function $\psi(\mathbf{r}, t)$. In general, the quantities $\int x^n|\psi|^2 dV$ appearing in these relations are different from the moments

$$\langle x^n\rangle = \int x^n \rho dV,$$

characterizing results of measurements. Requiring that relations (12) contain physically relevant moments $\langle x^n\rangle$ and

$$\int|\psi|^2 dV = 1,$$

we concluded that the most simple solution of these requirements has the form of the Born rule $\rho = |\psi|^2$ (see Section 2). At this point, the physical meaning of the phase s_1 of the probability amplitude

$$\psi = |\psi|\exp(is_1) = \exp(-s_2)\exp(is_1)$$

is not yet obvious.

In Section 3, we used the normalization condition

$$\int \rho dV = 1$$

and the Born rule $\rho = |\psi|^2$ and by applying the integration by parts, we obtained the commutation relation (25). This commutation relation only depends on s_2 and need not be considered if the theory does not have a probabilistic character, as, for example, in the case of classical mechanics. A similar commutation relation should appear in any probabilistic theory analogous to that described in this paper.

Subsequently, in Section 4, we started with the uncertainty relation (12) obtained in Section 2 and derived a few related uncertainty relations including the Heisenberg uncertainty relation in form (41) that does not contain the Planck constant $\hbar$. We found also the uncertainty relation (46) that does not depend on s_1 and its left-hand side is less than or equal to that in relation (41). Analogously to the commutation relations, the uncertainty relations should appear in any probabilistic theory analogous to that discussed in this paper. In the limit case in which the probabilistic character of the theory disappears, the commutation and uncertainty relations can be neglected.

The commutation relation (25) remains valid if the operator $-i(\partial/\partial x)$ is replaced by the operator $-i(\partial/\partial x) - f_x(\mathbf{r}, t)$, where f_x is an arbitrary real function. This makes it possible to generalize the theory and include, for example, the vector potential describing the interaction with the electromagnetic field (Section 7).

The physical meaning of the phase s_1 was discussed in Section 8. To describe the motion of particles in space, it is necessary to introduce not only the probability distribution ρ, but also the corresponding probability density current, which can be taken in the form $\mathbf{j} = \rho\mathbf{v}$. Here, the "velocity" can be written in the general form $\mathbf{v} = \hbar\nabla s_1/m_0$, where s_1 is the phase of the probability amplitude, m_0 is the rest mass of the particle, and $\hbar$ is a constant that has to be determined experimentally. Thus, the probability amplitude

$$\psi = \exp[i(s_1 + is_2)]$$

carries information on the probability density current $\mathbf{j}$ contained in s_1 and on the probability distribution ρ contained in $s_2 = -(1/2)\ln\rho$. The operator $-i(\partial/\partial x)$ appearing in the expression (50) for the probability density current is needed for describing the motion of particles in space.

The Fisher information characterizing the smoothness of the probability distribution ρ appears already in Eq. (12). Its generalization, including not only s_2, but also s_1, was discussed in Section 9. Except for a factor, this generalization of the Fisher information agrees with the kinetic energy known from quantum mechanics and shows its importance in physics. The momentum operator was discussed in Section 9 too.

To discuss time in Section 10, we assumed that the investigated system has a finite lifetime and the probability density can be normalized by the condition

$$\int\int_{t_0}^{\infty} \rho(\mathbf{r}, t) dV dt = 1.$$

Then, using an approach similar to that for the coordinate x, we introduced the time component of the probability density current j_t and the operator $i(\partial/\partial t)$ and derived the time–energy uncertainty relation (74). Different roles of time and the spatial coordinates in quantum mechanics were also discussed.

Analogously to the vector potential, we also discussed the scalar potential. The unidirectional character of time makes it possible to also understand the existence of antiparticles (Section 11).

Equations of motion are derived from the requirement that a linear combination of the space and time Fisher informations is a relativistic invariant (Section 12). It leads not only to the Klein–Gordon, Dirac, and Schrödinger equations, but it helps also to understand some assumptions made in special theory relativity.

The transition from quantum to classical mechanics is well-known. Therefore, only a brief discussion of the transition from the Schrödinger equation to the Hamilton–Jacobi equation of classical mechanics was made in Section 13.

Generalization to many-particle systems is straightforward and was made in Section 14.

The standard postulates of quantum mechanics and their relation to our approach were discussed in Section 15.

The discussion in the first part of this paper can be applied not only to the spatial coordinates x, y, and z, but also to some other physical quantities. However, it does not apply for the equations of motion that are derived from the requirement of the relativistic invariance of the space–time Fisher information $I_t''/c^2 - (I_x'' + I_y'' + I_z'')$ depending on x, y, z, and t.

As we mentioned in Section 2, our way of writing the probability amplitude in Eq. (20) is very similar to that used by Bohm [7, 8]. However, our main results and their interpretation are different from those of Bohm. Since there is no experimental evidence for a precisely defined position of particles assumed in [7, 8], we do not make such an assumption here. Also, we do not introduce any additional quantum potential like Bohm does, since such a potential would violate the commutation relations that are general consequence of the probabilistic description of measurements. In contrast to [7, 8], where the validity of the Schrödinger equation is postulated, we require here the relativistic invariance of the generalized space–time Fisher information leading to the relativistic equations of motion. We also note that no hidden variables are needed in our approach. Therefore, our understanding of quantum mechanics is close to the standard or Copenhagen interpretation and differs substantially from that of Bohm.

We also note that formulating the theory in terms of the probability amplitude ψ instead the original quantities—probability density ρ and the probability density current $\mathbf{j}$—has a very significant advantage, namely, the equations of motion are linear in ψ. Thus, the approach based on the probability amplitude leads to a mathematically simple theory based on the linear vector spaces in which the superposition principle for ψ is valid. If other quantities like ρ and $\mathbf{j}$ or s_1 and s_2 were used for representing the state of the system, this physically important property valid for ψ would be lost.

Another question is whether the relation between the probability amplitude ψ and the probability density ρ could be different from that given by the Born rule (17) or (18). For example, we can replace Eq. (18) by the equation

$$\xi = \rho^{1/4} e^{is_1/2}. \tag{131}$$

Then, we obtain the normalization condition

$$\int |\xi|^4 dV = 1 \tag{132}$$

and the mean square coordinate

$$\langle x^2 \rangle = \int x^2 |\xi|^4 dV. \tag{133}$$

Analogous formulas can also be obtained for other physical quantities by replacing the usual probability amplitude ψ by ξ^2. The resulting formulas that can be obtained from equations in Sections 2–15 are more complicated and the equations of motion are not linear in ξ. Therefore, the simplest theory with the probability amplitudes creating the linear vector spaces is obtained only for the usual relation between the probability density and probability amplitude in form of the Born rule.

Unperformed experiments have no results. Therefore, it follows from our discussion that quantum mechanics does not speak explicitly of events in the measured system, but only of the results of measurements, implying the existence of external measuring apparatuses that has been supposed above.

Of course, one can argue that some steps of our discussion were motivated by our knowledge of quantum mechanics. For this reason, the most important steps were discussed in detail and their relation to standard quantum mechanics was clarified. The resulting way of obtaining quantum mechanics is not unique. However, our discussion shows that quantum mechanics is, in the sense of Occam's razor, apparently the most simple and straightforward way of describing the probabilistic nature of certain class of physical phenomena with important physical and mathematical advantages. It shows also that the main ideas of quantum mechanics are understandable and physically and mathematically transparent.

Now we can return to Einstein's question of why the world is a quantum world. Taking into consideration that the interaction of the measured system with the measuring apparatus and the rest of the world cannot be in general neglected, measuring apparatuses are not described in detail, but on the macroscopic level only; real physical detectors have limited resolution and efficiency and the experimental control of the initial conditions is limited and probabilistic description of results of measurements seems to be unavoidable. As shown in this paper, such a probabilistic description of results of measurements then leads, together with the relativistic invariance of the theory and requirement of its mathematical and physical simplicity, to quantum mechanics. A deterministic description of the world in the sense of classical mechanics is possible only in special cases when the probabilistic character of measurements can be neglected.

Since the information theoretical approach used in this and other papers (see, e.g., [15–18, 22–24, 39, 42, 47, 48]) makes it possible to obtain the most significant parts of the mathematical formalism of quantum mechanics from the probabilistic description of results of measurements, we believe that this is a good starting point to understanding this field. This approach shows that the roots of quantum mechanics are deeply related to the probability theory. It also helps to understand quantum theory as correctly formulated probabilistic theory that can describe certain class of physical phenomena at different levels of accuracy from the simplest models to very complex ones.

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