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Two multi-dimensional uncertainty relations

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Abstract

Two multi-dimensional uncertainty relations, one related to the probability density and the other one related to the probability density current, are derived and discussed. Both relations are stronger than the usual uncertainty relations for the coordinates and momentum.

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1. Introduction

Uncertainty relations, one of the fundamental results of quantum mechanics, have been studied in a large number of papers (for a detailed review, see, e.g. [1]). The standard approach to their derivation is based on the wavefunction. However, it has been shown in [2–4] that a large part of the mathematical formalism of quantum mechanics can be obtained by generalizing the statistical analysis of experimental results with the corresponding probability distribution and probability density current. It is shown below that such an approach yields stronger uncertainty relations than the usual ones.

The aim of this paper is to find and discuss a multi-dimensional form of two uncertainty relations, one of which is related to the form of the probability density and the other one is related to the probability density current.

2. First uncertainty relation

We consider the N-dimensional space with the coordinates

$$\mathbf{x} = (x_1, \ldots, x_N) \tag{1}$$

and the probability density $\rho \geqslant 0$ given by the wavefunction ψ

$$\rho(\mathbf{x}) = |\psi(\mathbf{x})|^2. \tag{2}$$

It is assumed that this probability density fulfils the boundary conditions

$$\rho|_{x_m=-\infty}^{\infty} = 0, \qquad m = 1, \ldots, N \tag{3}$$

and the standard normalization condition

$$\int \rho(\mathbf{x}) \mathrm{d}\tau = 1, \qquad \mathrm{d}\tau = \mathrm{d}x_1, \ldots, \mathrm{d}x_N, \tag{4}$$

where the integration is carried out over the whole space.

The mean values of the coordinates are defined as

$$\langle x_m \rangle = \int x_m \rho \, \mathrm{d}\tau. \tag{5}$$

The $N \times N$ covariance matrix $(\Delta X)^2$ is given by the equation

$$(\Delta X)^2_{mn} = \int (x_m - \langle x_m \rangle)(x_n - \langle x_n \rangle) \rho \, \mathrm{d}\tau. \tag{6}$$

Assuming that c_m are arbitrary complex numbers it can easily be verified that this matrix is positive semidefinite

$$\sum_{m,n=1}^{N} c_m^* (\Delta X)^2_{mn} c_n = \int \left| \sum_{m=1}^{N} c_m (x_m - \langle x_m \rangle) \right|^2 \rho \, \mathrm{d}\tau \geqslant 0, \tag{7}$$

where the star denotes the complex conjugate.

In the following discussion, the wavefunction ψ will be written in the form (see, e.g. [2–4])

$$\psi = \mathrm{e}^{(\mathrm{i}s_1 - s_2)/\hbar}, \tag{8}$$

where $s_1 = s_1(x_1, \ldots, x_N)$ and $s_2 = s_2(x_1, \ldots, x_N)$ are real functions. The functions s_1 and s_2 give the probability density ρ

$$\rho = |\psi|^2 = \mathrm{e}^{-2s_2/\hbar} \tag{9}$$

and the probability density current

$$j_k = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x_k} - \psi \frac{\partial \psi^*}{\partial x_k} \right) = \frac{1}{m} \frac{\partial s_1}{\partial x_k} \rho. \tag{10}$$

Further, we calculate the mean value of the momentum operator $\hat{p}_m = -\mathrm{i}\hbar(\partial/\partial x_m)$ which must be real

$$\langle \hat{p}_m \rangle = \int \psi^* \hat{p}_m \psi \, \mathrm{d}\tau = \int \frac{\partial s_1}{\partial x_m} \rho \, \mathrm{d}\tau + \mathrm{i} \int \frac{\partial s_2}{\partial x_m} \rho \, \mathrm{d}\tau = \int \frac{\partial s_1}{\partial x_m} \rho \, \mathrm{d}\tau = \left\langle \frac{\partial s_1}{\partial x_m} \right\rangle. \tag{11}$$

Similarly, we get

$$\begin{aligned} \langle \hat{p}_m \hat{p}_n \rangle = \int (\hat{p}_m \psi)^* (\hat{p}_n \psi) \, \mathrm{d}\tau &= \int \left(\frac{\partial s_1}{\partial x_m} \frac{\partial s_1}{\partial x_n} + \frac{\partial s_2}{\partial x_m} \frac{\partial s_2}{\partial x_n} \right) \rho \, \mathrm{d}\tau \\ &\quad + \mathrm{i} \int \left(\frac{\partial s_1}{\partial x_m} \frac{\partial s_2}{\partial x_n} - \frac{\partial s_2}{\partial x_m} \frac{\partial s_1}{\partial x_n} \right) \rho \, \mathrm{d}\tau \\ &= \int \left(\frac{\partial s_1}{\partial x_m} \frac{\partial s_1}{\partial x_n} + \frac{\partial s_2}{\partial x_m} \frac{\partial s_2}{\partial x_n} \right) \rho \, \mathrm{d}\tau \end{aligned} \tag{12}$$

and

$$\begin{aligned} (\Delta P)^2_{mn} &= \int [(\hat{p}_m - \langle \hat{p}_m \rangle) \psi]^* (\hat{p}_n - \langle \hat{p}_n \rangle) \psi \, \mathrm{d}\tau = \langle \hat{p}_m \hat{p}_n \rangle - \langle \hat{p}_m \rangle \langle \hat{p}_n \rangle \\ &= \int \left(\frac{\partial s_1}{\partial x_m} \frac{\partial s_1}{\partial x_n} + \frac{\partial s_2}{\partial x_m} \frac{\partial s_2}{\partial x_n} \right) \rho \, \mathrm{d}\tau - \int \frac{\partial s_1}{\partial x_m} \rho \, \mathrm{d}\tau \int \frac{\partial s_1}{\partial x_n} \rho \, \mathrm{d}\tau. \end{aligned} \tag{13}$$

Analogously to the matrix $(\Delta X)^2$, it can be shown that the matrix $(\Delta P)^2$ is positive semidefinite.

It can be verified that both matrices appearing in equation (13)

$$(\Delta P_1)^2_{mn} = \int \frac{\partial s_1}{\partial x_m}\frac{\partial s_1}{\partial x_n}\rho\,\mathrm{d}\tau - \int \frac{\partial s_1}{\partial x_m}\rho\,\mathrm{d}\tau \int \frac{\partial s_1}{\partial x_n}\rho\,\mathrm{d}\tau \tag{14}$$

and

$$(\Delta P_2)^2_{mn} = \int \frac{\partial s_2}{\partial x_m}\frac{\partial s_2}{\partial x_n}\rho\,\mathrm{d}\tau, \tag{15}$$

$$(\Delta P_1)^2 + (\Delta P_2)^2 = (\Delta P)^2 \tag{16}$$

are positive semidefinite, too. The matrix $(\Delta P_1)^2$ depends on $(\partial s_1/\partial x_m)$ and ρ (i.e., s_2) and can be different from zero for the nonzero probability density current $\mathbf{j} = (j_1, \ldots, j_N)$ only (see equation (10)). The second matrix $(\Delta P_2)^2$ depends only on the form of the wave packet given by s_2 and is independent of $\mathbf{j}$. Therefore, the separation of $(\Delta P)^2$ into two parts given by equations (14)–(16) has a good physical meaning. An analogous separation was discussed also in [5] within the framework of the one-dimensional stochastic mechanics.

Now, we define a correlation matrix G

$$G_{mn} = \int (x_m - \langle x_m\rangle)\left(\frac{\partial s_1}{\partial x_n} - \left\langle\frac{\partial s_1}{\partial x_n}\right\rangle\right)\rho\,\mathrm{d}\tau \tag{17}$$

and create a new $2N \times 2N$ matrix M

$$M = \begin{pmatrix} (\Delta X)^2 & G^T \\ G & (\Delta P_1)^2 \end{pmatrix}, \tag{18}$$

where the superscript T denotes the transposition. To show that the matrix M is also positive semidefinite we define the quantities f_m

$$f_m = x_m - \langle x_m\rangle, \qquad f_{N+m} = \frac{\partial s_1}{\partial x_m} - \left\langle\frac{\partial s_1}{\partial x_m}\right\rangle, \qquad m = 1, \ldots, N. \tag{19}$$

It follows from

$$\sum_{m,n=1}^{2N} c_m^* M_{mn} c_n = \int \left|\sum_{m=1}^{2N} c_m f_m\right|^2 \rho\,\mathrm{d}\tau \geqslant 0 \tag{20}$$

that the matrix M is positive semidefinite, too. Further, we make use of a general result valid for $N \times N$ matrices A, B, C and D, where D is a regular matrix

$$\begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix} \tag{21}$$

leading to

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D). \tag{22}$$

Applying this equation to the matrix M given by equation (18), the first multi-dimensional uncertainty relation for the matrices $(\Delta X)^2$ and $(\Delta P_1)^2$ is obtained

$$\det(M) = \det\{(\Delta X)^2(\Delta P_1)^2 - G^T[(\Delta P_1)^2]^{-1}G(\Delta P_1)^2\} \geqslant 0. \tag{23}$$

In the one-dimensional form, this relation can be written as

$$\int (x - \langle x\rangle)^2\rho\,\mathrm{d}x \int \left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)^2 \rho\,\mathrm{d}x \geqslant \left(\int (x - \langle x\rangle)\left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)\rho\,\mathrm{d}x\right)^2. \tag{24}$$

We note that the last inequality is nothing but the Schwarz inequality $(u,u)(v,v) \geqslant |(u,v)|^2$ for the functions $u = (x - \langle x\rangle)\sqrt{\rho}$, $v = (\partial s_1/\partial x - \langle \partial s_1/\partial x\rangle)\sqrt{\rho}$ and the inner product (u,v) is defined by means of the integration over all x. Equation (24) was also derived within the framework of Nelson's stochastic mechanics [5].

3. Second uncertainty relation

Further, we replace the function s_1 by s_2 and repeat discussion made in the preceding paragraph. Taking into account the equation

$$\left\langle \frac{\partial s_2}{\partial x_m}\right\rangle = \int \frac{\partial s_2}{\partial x_m}\rho\,\mathrm{d}\tau = -\frac{\hbar}{2}\int \frac{\partial \rho}{\partial x_m}\,\mathrm{d}\tau = -\frac{\hbar}{2}\int \rho|_{x_m=-\infty}^{\infty}\,\mathrm{d}\tau' = 0, \tag{25}$$

where $\mathrm{d}\tau' = \mathrm{d}x_1,\ldots,\mathrm{d}x_{m-1}\,\mathrm{d}x_{m+1},\ldots,\mathrm{d}x_N$, the matrix element G_{mm} can be replaced by G'_{mm}

$$G'_{mm} = \int (x_m - \langle x_m\rangle)\frac{\partial s_2}{\partial x_m}\rho\,\mathrm{d}\tau. \tag{26}$$

Performing here the integration by parts in the variable x_m, assuming that $[(x_m - \langle x_m\rangle)\rho]_{x_m=-\infty}^{\infty} = 0$ and using equations (4)–(5) we get

$$G'_{mm} = -\frac{\hbar}{2}\int [(x_m - \langle x_m\rangle)\rho]_{x_m=-\infty}^{\infty}\,\mathrm{d}\tau' + \frac{\hbar}{2}\int \rho\,\mathrm{d}\tau = \frac{\hbar}{2}. \tag{27}$$

By using equations (3)–(5) we get similarly

$$G'_{mn} = \int (x_m - \langle x_m\rangle)\frac{\partial \rho}{\partial x_n}\,\mathrm{d}\tau = \int (x_m - \langle x_m\rangle)\rho|_{x_n=-\infty}^{\infty}\,\mathrm{d}\tau'' = 0, \qquad m \neq n, \tag{28}$$

where $\mathrm{d}\tau'' = \mathrm{d}x_1,\ldots,\mathrm{d}x_{n-1}\,\mathrm{d}x_{n+1},\ldots,\mathrm{d}x_N$. Equation (22) applied to the matrix

$$M' = \begin{pmatrix} (\Delta X)^2 & \hbar/2 \\ \hbar/2 & (\Delta P_2)^2 \end{pmatrix} \tag{29}$$

then yields the second multi-dimensional uncertainty relation

$$\det\left[(\Delta X)^2(\Delta P_2)^2 - \frac{\hbar^2}{4}\right] \geqslant 0 \tag{30}$$

or in the one-dimensional form

$$\int (x - \langle x\rangle)^2\rho\,\mathrm{d}x \int \left(\frac{\partial s_2}{\partial x}\right)^2 \rho\,\mathrm{d}x \geqslant \frac{\hbar^2}{4}. \tag{31}$$

Due to equations (26)–(27), the last inequality is the Schwarz inequality for $u = (x - \langle x\rangle)\sqrt{\rho}$ and $v = (\partial s_2/\partial x)\sqrt{\rho}$. Equation (31) is known for example from [6], see also the stochastic variational approach to the minimum uncertainty states [7]. Another discussion of equation (31) can be found in [2–4].

4. Conclusions

The last uncertainty relation can be obtained by replacing $(\Delta P_2)^2$ in the matrix M' by $(\Delta P)^2 = (\Delta P_1)^2 + (\Delta P_2)^2$. The resulting matrix remains positive semidefinite, and equation (30) then leads to the multi-dimensional uncertainty relation

$$\det\left[(\Delta X)^2(\Delta P)^2 - \frac{\hbar^2}{4}\right] \geqslant 0 \tag{32}$$

which is known for example from [1, 9, 10, 13, 14]. The one-dimensional form of this relation is the Heisenberg uncertainty relation (see [11, 12])

$$\langle(\Delta x)^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \frac{\hbar^2}{4}. \tag{33}$$

There is one important difference between the usual uncertainty relations (32)–(33) and the uncertainty relations (30)–(31). The usual uncertainty relations are based on using the wavefunction ψ. In contrast to it, the uncertainty relations (30)–(31) use $s_2 = -(\hbar/2)\ln\rho$, i.e. the probability density ρ only. Due to equation (16), the uncertainty relations (30)–(31) are in general stronger than the usual uncertainty relations. Similar arguments apply for the uncertainty relations (23)–(24), too.

It is worth noting that the matrix

$$I_{mn} = \int \frac{1}{\rho}\frac{\partial\rho}{\partial x_m}\frac{\partial\rho}{\partial x_n}\,\mathrm{d}\tau = 4(\Delta P_2)^2_{mn}/\hbar^2 \tag{34}$$

can be interpreted as the Fisher information matrix. The Fisher information in the one-dimensional form (see, e.g. [15–23])

$$I = \int \frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2 \mathrm{d}x \tag{35}$$

and its matrix analogy are important quantities in statistical mathematics. They appear for example in the Rao–Cramér inequalities [8, 16] playing a similar role in mathematical statistics as the uncertainty relations in quantum physics.

Concluding, we see that the uncertainty relations (32)–(33) are in our approach replaced by more detailed equations (23)–(24) and (30)–(31) for the information carried by the functions s_1 and s_2, i.e. information related to the probability density current $\mathbf{j}$ and the probability density ρ. For real wavefunctions ψ, corresponding to $s_1 = 0$, equation (24) gives trivial result $0 = 0$, and equation (31) becomes equation (33). In a general case, equations (23)–(24) and (30)–(31) are stronger uncertainty relations than equations (32)–(33). From this point of view, equations (23)–(24) and (30)–(31) are preferable to equations (32)–(33).

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