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Quantum Mechanics and Statistical Description of Results of Measurement

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1. Introduction

Quantum mechanics and its meaning have been discussed in a large number of publications from many different points of view (see e.g. books (Auletta, 2001; Wheeler & Zurek, 1981)). It shows that quantum mechanics is, despite its successful applications, difficult to understand. In this chapter, we discuss quantum mechanics from the point of view of mathematical statistics and show that the most important parts of the mathematical formalism of quantum mechanics can be derived from the statistical description of results of measurement. Various aspects of this approach can be found for example in (Frieden, 1998; 2004; Frieden & Soffer, 1995; Kapsa & Skála, 2009; 2011; Kapsa et al., 2010; Reginatto, 1998; 1999; Skála & Kapsa, 2005a;b; 2007a;b; 2011; Skála, Čížek & Kapsa, 2011).

One of the main differences between classical and quantum mechanics is consistent statistical description of results of measurement in quantum mechanics. In contrast to classical mechanics according to which physical measurement can be made in principle arbitrarily exact, quantum mechanics takes into consideration physical reality confirmed by experiments and describes physical measurement statistically. The most important points of the statistical description of measurement of the space coordinate x are summarized in Section 2. An important quantity appearing in this approach is the probability density $\rho(x,t)$ of obtaining the value x in measurement made at time t. For the sake of simplicity, only one spatial coordinate x is taken here.

Due to the normalization condition for the probability density corresponding to the fact that the measured system must be somewhere in space the probability density ρ must obey the continuity equation analogous to that known from classical continuum mechanics. Therefore, except for ρ, we have to take into account also the corresponding probability density current $j(x,t)$ appearing in the continuity equation. We note that the density current j is also necessary for describing the motion in space. To describe the statistical state of the system, both quantities ρ and j are necessary. It is shown in Section 3 that instead of two real quantities ρ and j, we can use also two real functions $s_1(x,t)$ and $s_2(x,t)$ given by equations $\rho = \exp(-2s_2/\hbar)$ and $j = \rho\, v = \rho p/m = \rho(\partial s_1/\partial x)/m$, where s_1 corresponds to the Hamilton action S in the expression $p = \partial S/\partial x$ known from the Hamilton–Jacobi theory of classical mechanics. More compact way of describing the statistical state of the system is to use the complex wave function $\psi = \exp[(is_1 - s_2)/\hbar]$ as it is done in quantum mechanics. We note that the expression for the probability density current $j = \rho(\partial s_1/\partial x)/m$ is equivalent to the expression for the probability density current known from quantum mechanics.

By analogy with the expression for the momentum $p = \partial S/\partial x$ one can make an attempt to represent the momentum by the function $p = \partial s_1/\partial x$. It is shown in Sections 4 and 5 that in case of the mean momentum $\langle p \rangle$ and the mean value $\langle xp \rangle$ this definition gives the same results as the quantum–mechanical representation of the momentum $\hat{p} = -i\hbar(\partial/\partial x)$. However, it is not true in more complicated cases when the operator representation of the momentum has to be used.

One of important quantities appearing in mathematical statistics is the Fisher information. It is shown in Section 6 that the Fisher information $I_x = (4/\hbar^2)\int_{-\infty}^{\infty}(\partial s_2/\partial x)^2 \rho \,\mathrm{d}x$ fulfills the inequality $\langle (x-a)^2 \rangle I_x \geq 1$, where a is a real constant. This inequality is analogous to the uncertainty relations known from quantum mechanics and is general property of statistical theories similar to that used in quantum mechanics.

It is shown in Section 7 that the kinetic energy in quantum mechanics can be written as a sum of two terms. The first term is statistical generalization of the kinetic energy known from classical mechanics. The second part of the kinetic energy is proportional to the Fisher information I_x and does not have its counterpart in classical mechanics. Therefore, in contrast to classical mechanics, the Fisher information is an important part of the kinetic energy in quantum mechanics.

Similarly to the kinetic energy, the mean value $\langle (\Delta p)^2 \rangle$ appearing in the Heisenberg uncertainty relation $\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \hbar^2/4$ can be written as a sum of two terms $\langle (\Delta p)^2 \rangle = \langle (\Delta p_1)^2 \rangle + \langle (\Delta p_2)^2 \rangle$ (Section 8). Again, the first term can be understood as statistical generalization of the expression known from classical mechanics. The second term is proportional to the Fisher information I_x. If the first term equals zero, the Heisenberg uncertainty relation is equivalent to the inequality for the Fisher information mentioned above. It shows that the inequality for the Fisher information is in quantum mechanics correctly respected.

It is shown in Section 9 that the Heisenberg uncertainty relation can be replaced by two stronger uncertainty relations for $\langle (\Delta p_1)^2 \rangle$ and $\langle (\Delta p_2)^2 \rangle$. The sum of these two uncertainty relations is equivalent to the Robertson–Schrödinger uncertainty relation (Section 10). By neglecting one term at the right–hand side of the Robertson–Schrödinger uncertainty relation the Heisenberg uncertainty relation is obtained. Therefore, two uncertainty relations discussed in Section 9 are stronger than the corresponding Heisenberg and Robertson–Schrödinger uncertainty relations. It is worth noting that the second uncertainty relation equivalent to the inequality for the Fisher information depends only on the function s_2 or the envelop of the wave function $|\psi|$. Since it does not depend on s_1, inequality in this relation can be achieved for much larger class of the wave functions than in case of the Heisenberg and Robertson–Schrödinger uncertainty relations. It may be important in some applications as for example in the theory of the most efficient information transfer.

Two examples illustrating results of Sections 8–10, namely the gaussian wave packet for a free particle and the linear harmonic oscillator are discussed in Sections 11 and 12.

By using the normalization condition for $\rho = |\psi|^2$ it is possible to derive the equation indicating validity of the commutation relation $[x, \hat{p}] = i\hbar$ (Section 13). This commutation relation shows that it is possible to replace the momentum operator $\hat{p}$ by the operator $\hat{p} - f$, where $f(x,t)$ is a real function. This function can describe external conditions in which the system moves and corresponds to the x-component of the vector potential.

In standard quantum mechanics, systems with the infinite lifetime are usually considered. In such a case, the normalization condition for the probability density $\int_{-\infty}^{\infty} \rho \,\mathrm{d}x = 1$ is valid at all times and it does make sense to introduce the probability density in time analogous to

the probability density in space. For this reason, time is taken as a parameter in standard quantum mechanics. In Section 14, systems with a finite lifetime are considered and a decaying probability to find the system anywhere in space $\nu(t) = \int_{-\infty}^{\infty} \rho \, \mathrm{d}x$ is introduced. It makes possible to define the mean lifetime and other quantities by analogy with those for the coordinate x.
Similarly to Section 13, it is then possible to get the commutation relation for the operator $i\hbar(\partial/\partial t)$ and time t and to find mathematical arguments for the existence of the scalar potentials (Section 15).
For systems with exponentially decaying wave functions, it possible to derive also the time–energy uncertainty relations (Section 16).
Equations of motion are discussed in Section 17. To derive the equation of motion, the Fisher information I_x defined for the space coordinate x is first generalized to two Fisher informations J_x and J_t in space–time in which the derivatives of the functions s_1 and s_2 with respect to x and t are taken into account. Then, the combined space–time Fisher information $J_t/c^2 \pm J_x$ is discussed. Further, we require that our theory is independent of the choice of the coordinate system in space–time and the concrete initial conditions. It yields the equation $J_t/c^2 - J_x = const$, where the signs of the space and time parts are different similarly to the signs in the metric in special relativity and $const \geq 0$. Formulating this condition in the variational form, it leads to the equation of motion mathematically equivalent to the Klein–Gordon equation. The Schrödinger equation can be viewed as the non–relativistic approximation to the Klein–Gordon equation. The Dirac equation can be obtained in a similar way. It is shown also that the equations of motion in quantum mechanics should be linear.

2. Statistical description of results of measurement

In this section, we discuss probably the most important difference between classical and quantum mechanics — statistical description of results of measurement.
We note that the measuring apparatus is not described in quantum mechanics on the microscopic level and the measured system interacts with the measuring apparatus. For this reason, results of measurement have to be described statistically. In agreement with experimental experience, we assume that results of repeated measurement of the coordinate x can be characterized by the mean values

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x,t) \, \mathrm{d}x, \tag{1}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\rho(x,t) \, \mathrm{d}x \tag{2}$$

and the corresponding mean square displacement

$$\langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2. \tag{3}$$

Here,

$$\Delta x = x - \langle x \rangle \tag{4}$$

an $\rho(x,t) \geq 0$ is a normalized probability density giving the probability of obtaining the value x in measurement at time t

$$\int_{-\infty}^{\infty} \rho \, \mathrm{d}x = 1. \tag{5}$$

For the sake of simplicity, we assume that $\rho(x,t)$ fulfills the boundary conditions

$$\lim_{x\to\pm\infty} x^n \rho = 0, \quad n = 0,1,2. \tag{6}$$

We assume also that in the limit of classical mechanics

$$\rho(x,t) \to \delta(x - x_{cl}) \tag{7}$$

the mean coordinate $\langle x \rangle$ converges to the classical coordinate $x_{cl} = x_{cl}(t)$.

3. Wave function

From the point of view of our statistical description, the wave function ψ can be introduced in the following simple way.
First, we introduce a real function $s_2 = s_2(x,t)$ by the equation

$$\rho = \mathrm{e}^{-2s_2/\hbar} \tag{8}$$

or equivalently

$$s_2 = -\frac{\hbar}{2} \ln \rho, \tag{9}$$

where $\hbar$ denotes the reduced Planck constant, $\hbar = h/(2\pi)$. We note that the transition $\rho(x,t) \to \delta(x - x_{cl})$ can be formally performed for $\hbar \to 0_+$.
Due to normalization condition (5), the probability density ρ has to obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0, \tag{10}$$

where $j = j(x,t)$ is the probability density current in one dimension and $\partial j/\partial x$ is the divergence in one dimension.
Analogously to continuum mechanics, it is possible to express the probability density current j in terms of the "velocity" v

$$j = \rho\, v. \tag{11}$$

Further, by analogy with the expression $v = p/m = (\partial S/\partial x)/m$ from the Hamilton–Jacobi theory we can write

$$v = \frac{\partial s_1/\partial x}{m}, \tag{12}$$

where m is the mass of the system, a real function $s_1 = s_1(x,t)$ corresponds to the Hamilton action $S = S(x,t)$ and the function $\partial s_1/\partial x$ represents the momentum in our statistical approach. In the limit of classical mechanics when the statistical description disappears, the function s_1 has to fulfill the condition $s_1(x,t) \to S(x_{cl},t)$ and $\partial s_1/\partial x \to \partial S/\partial x$.
It is seen that instead of two quantities ρ and j, the statistical state of the system can be described by two mutually independent real functions s_1 and s_2 or a new complex function ψ

$$\psi = \mathrm{e}^{(is_1 - s_2)/\hbar} \tag{13}$$

depending on s_1 and s_2 (see also (Madelung, 1926)). Using this function, the probability density ρ and probability density current j given above can be rewritten in the form known from quantum mechanics

$$\rho = |\psi|^2 \tag{14}$$

and

$$j = \rho \frac{\partial s_1 / \partial x}{m} = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right), \tag{15}$$

where the star denotes the complex conjugate.
The function ψ called the wave function in quantum mechanics is only a different way of representing the statistical state of the system described by two real functions ρ and j or s_1 and s_2.
We note that our expression for the wave function (13) is similar to that of Bohm (Bohm, 1952a;b). However, we do not assume the existence of hidden variables here.

4. Momentum operator

By analogy with Eq. (1) and our discussion in the preceding section, the mean momentum can be defined as (see also (Skála, Čížek & Kapsa, 2011))

$$\langle p \rangle = \int_{-\infty}^{\infty} \frac{\partial s_1}{\partial x} \rho \, \mathrm{d}x. \tag{16}$$

It follows from conditions (6) that the integral

$$\int_{-\infty}^{\infty} \frac{\partial s_2}{\partial x} \rho \, \mathrm{d}x = -\frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial x} \mathrm{d}x = -\frac{\hbar}{2} \rho |_{x=-\infty}^{\infty} = 0 \tag{17}$$

equals zero. Using this result it is easy to verify that Eq. (16) can be also written as

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \, \psi \, \mathrm{d}x, \tag{18}$$

where the momentum operator equals

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}. \tag{19}$$

Equations (16) and (18) yield the same result and representation of the momentum by the function $\partial s_1 / \partial x$ and the operator $\hat{p}$ is in this case equivalent.

5. Mean value of xp

In this section, we investigate the mean value of the product of the coordinate and momentum which is important in the uncertainty relations (see also (Skála, Čížek & Kapsa, 2011)).
As it is known, the mean value of the product of the coordinate and momentum is in quantum mechanics given by the expression

$$\frac{\langle x\hat{p} \rangle + \langle \hat{p}x \rangle}{2} = \frac{1}{2} \int_{-\infty}^{\infty} \psi^* \left[x \left(-i\hbar \frac{\partial}{\partial x} \right) + \left(-i\hbar \frac{\partial}{\partial x} \right) x \right] \psi \, \mathrm{d}x. \tag{20}$$

Using Eq. (13) we get

$$\frac{\langle x\hat{p} \rangle + \langle \hat{p}x \rangle}{2} = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{(-is_1 - s_2)/\hbar} \left[2x \left(-i\hbar \frac{\partial}{\partial x} \right) - i\hbar \right] \mathrm{e}^{(is_1 - s_2)/\hbar} \mathrm{d}x. \tag{21}$$

Now we calculate the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{(-is_1-s_2)/\hbar} x\Big(-i\hbar\frac{\partial}{\partial x}\Big)\mathrm{e}^{(is_1-s_2)/\hbar}\mathrm{d}x = \int_{-\infty}^{\infty} x\frac{\partial s_1}{\partial x}\rho\,\mathrm{d}x + i\int_{-\infty}^{\infty} x\frac{\partial s_2}{\partial x}\rho\,\mathrm{d}x. \tag{22}$$

By using integration by parts in the last integral and Eqs. (5) and (6) we get

$$\int_{-\infty}^{\infty} x\frac{\partial s_2}{\partial x}\rho\,\mathrm{d}x = x\frac{-\hbar}{2}\rho|_{x=-\infty}^{\infty} + \frac{\hbar}{2}\int_{-\infty}^{\infty}\rho\,\mathrm{d}x = \frac{\hbar}{2}. \tag{23}$$

The resulting formula

$$\frac{\langle x\hat{p}\rangle + \langle \hat{p}x\rangle}{2} = \int_{-\infty}^{\infty} x\frac{\partial s_1}{\partial x}\rho\,\mathrm{d}x \tag{24}$$

agrees with the expression

$$\langle xp\rangle = \int_{-\infty}^{\infty} x\frac{\partial s_1}{\partial x}\rho\,\mathrm{d}x \tag{25}$$

analogous to Eqs. (1) and (16).

Summarizing results of the last two sections we see that contribution of the function $\partial s_2/\partial x$ to the mean values $\langle\hat{p}\rangle$ and $(\langle x\hat{p}\rangle + \langle\hat{p}x\rangle)/2$ equals zero and the momentum operator can be in these cases represented either by the function $p = \partial s_1/\partial x$ or the operator $\hat{p} = -i\hbar(\partial/\partial x)$. However, as it will be seen in the following sections, it it not true in more complicated cases.

6. Fisher information

The Fisher information is a very important quantity appearing in mathematical statistics (see e.g. (Cover & Thomas, 1991; Fisher, 1925)). In our case, it can be introduced in the following simple way (see also (Frieden, 1998; 2004; Frieden & Soffer, 1995; Kapsa & Skála, 2009; 2011; Kapsa et al., 2010; Reginatto, 1998; 1999; Skála & Kapsa, 2005a;b; 2007a;b; 2011; Skála, Čížek & Kapsa, 2011)). For various applications of the Fisher information in physics and chemistry see e.g. (Chakrabarty, 2004; Hornyák & Nagy, 2007; Nagy, 2003; 2006; 2007; Nagy & Liu, 2008; Nagy & Sen, 2006; Romera & Nagy, 2008; Szabó et al., 2008).

We start with normalization condition (5) for the probability density ρ in which we perform integration by parts

$$\big[(x-a)\rho\big]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}(x-a)\frac{\partial\rho}{\partial x}\,\mathrm{d}x = 1, \tag{26}$$

where a is an arbitrary real number. Taking into account Eq. (6) we get the starting point of the following discussion

$$\int_{-\infty}^{\infty}(x-a)\frac{\partial\rho}{\partial x}\,\mathrm{d}x = -1. \tag{27}$$

Now we make use of the Schwarz inequality for the inner product $(u,v) = \int_{-\infty}^{\infty} u^* v\,\mathrm{d}x$ of two complex functions u and v

$$(u,u)(v,v) \geq |(u,v)|^2. \tag{28}$$

Putting

$$u = (x-a)\sqrt{\rho}, \tag{29}$$

$$v = \frac{1}{\sqrt{\rho}}\frac{\partial\rho}{\partial x} \tag{30}$$

in inequality (28) and using Eq. (27) we get

$$\int_{-\infty}^{\infty}(x-a)^2\rho\,\mathrm{d}x\int_{-\infty}^{\infty}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\mathrm{d}x\geq 1. \tag{31}$$

Here, the second integral is the well–known quantity from mathematical statistics called the Fisher information

$$I_x=\int_{-\infty}^{\infty}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\mathrm{d}x\geq 0. \tag{32}$$

Inequality (31) is usually written in the form (Fisher, 1925)

$$\langle(x-a)^2\rangle\ I_x\geq 1. \tag{33}$$

This result is very general and does not depend on the concrete meaning of the variable x. Interpretation of the last inequality is similar to that of the uncertainty relations in quantum mechanics: For given I_x the integral $\langle(x-a)^2\rangle$ cannot be smaller than $1/I_x$ and vice versa. The minimum of the integral $\langle(x-a)^2\rangle$ is obtained for $a=\langle x\rangle$.
We note that inequality (33) in a more general form is known in mathematical statistics as the Rao–Cramér inequality (Cover & Thomas, 1991; Cramér, 1946a;b; Rao, 1945; 1992). Hence, any correctly formulated statistical theory has to lead to inequality (33) or an analogous one.
Using Eq. (8) for the probability density the Fisher information can be written in the equivalent form

$$I_x=\frac{4}{\hbar^2}\int_{-\infty}^{\infty}\left(\frac{\partial s_2}{\partial x}\right)^2\rho\,\mathrm{d}x=\frac{4}{\hbar^2}\left\langle\left(\frac{\partial s_2}{\partial x}\right)^2\right\rangle \tag{34}$$

which will appear in the following discussion.

7. Kinetic energy

Now we discuss the kinetic energy T in quantum mechanics

$$T=\int_{-\infty}^{\infty}\frac{|(\hat{p}-qA)\psi|^2}{2m}\mathrm{d}x, \tag{35}$$

where q denotes the charge, m the mass and A is the vector potential in one dimension (see also (Skála, Čížek & Kapsa, 2011)).
Using Eq. (13) for the wave function and Eq. (19) for the momentum operator we get

$$(\hat{p}-qA)\psi=\left(\frac{\partial s_1}{\partial x}+i\frac{\partial s_2}{\partial x}-qA\right)\mathrm{e}^{(is_1-s_2)/\hbar} \tag{36}$$

and

$$|(\hat{p}-qA)\psi|^2=\left[\left(\frac{\partial s_1}{\partial x}-qA\right)^2+\left(\frac{\partial s_2}{\partial x}\right)^2\right]\rho. \tag{37}$$

Therefore, the kinetic energy

$$T=\int_{-\infty}^{\infty}\frac{(\partial s_1/\partial x-qA)^2+(\partial s_2/\partial x)^2}{2m}\rho\,\mathrm{d}x \tag{38}$$

can be written as a sum of two terms

$$T = T_1 + T_2, \tag{39}$$

where

$$T_1 = \int_{-\infty}^{\infty} \frac{(\partial s_1/\partial x - qA)^2}{2m} \rho \, \mathrm{d}x \tag{40}$$

and

$$T_2 = \int_{-\infty}^{\infty} \frac{(\partial s_2/\partial x)^2}{2m} \rho = \frac{\hbar^2 I_x}{8m}. \tag{41}$$

The first term T_1 is statistical generalization of the kinetic energy known from classical mechanics. The second part of the kinetic energy T_2 depending on $\partial s_2/\partial x$ is proportional to the Fisher information I_x and does not have its counterpart in classical mechanics. Therefore, in contrast to classical mechanics, the Fisher information is an important part of the kinetic energy in quantum mechanics.

8. Heisenberg uncertainty relations

For the sake of simplicity, we assume that the potential A equals zero.
The Heisenberg uncertainty relation (Heisenberg, 1927) for the coordinate x and momentum p has the form

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{\hbar^2}{4}, \tag{42}$$

where

$$\langle (\Delta x)^2 \rangle = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 |\psi|^2 \mathrm{d}x \tag{43}$$

and

$$\langle (\Delta p)^2 \rangle = \int_{-\infty}^{\infty} \left| \left(-i\hbar \frac{\partial}{\partial x} - \langle \hat{p} \rangle \right) \psi \right|^2 \mathrm{d}x. \tag{44}$$

Using Eqs. (13), (14) and (17) we get

$$\langle (\Delta p)^2 \rangle = \langle (\Delta p_1)^2 \rangle + \langle (\Delta p_2)^2 \rangle, \tag{45}$$

where

$$\langle (\Delta p_1)^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right)^2 \rho \, \mathrm{d}x \tag{46}$$

and

$$\langle (\Delta p_2)^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\partial s_2}{\partial x} \right)^2 \rho \, \mathrm{d}x = \frac{\hbar^2}{4} I_x. \tag{47}$$

We see that, analogously to the kinetic energy T, the mean square deviation of the momentum $\langle (\Delta p)^2 \rangle$ can be split into two parts (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2011; Skála, Čížek & Kapsa, 2011)).
The first part $\langle (\Delta p_1)^2 \rangle$ corresponds to the representation of the momentum by the function $p = \partial s_1/\partial x$ and the first part of the kinetic energy T_1.
The second part $\langle (\Delta p_2)^2 \rangle$ is proportional to the Fisher information I_x and corresponds to the second part of the kinetic energy T_2. We note that for $\langle (\Delta p_1)^2 \rangle = 0$, the Heisenberg uncertainty relation (42) has the form of inequality (33) for the Fisher information with $a = \langle x \rangle$ (see also (Chakrabarty, 2004; Kapsa & Skála, 2009; 2011; Kapsa et al., 2010; Skála & Kapsa, 2005a;b; 2007a;b; 2011; Skála, Čížek & Kapsa, 2011)). Therefore, inequality (33) is in quantum mechanics correctly respected.

9. Two uncertainty relations

It is shown in this section that the Heisenberg uncertainty relation can be replaced by two uncertainty relations for $\langle(\Delta p_1)^2\rangle$ and $\langle(\Delta p_2)^2\rangle$ (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2008; 2009; 2011; Skála, Čížek & Kapsa, 2011)).

According to the well–known result of mathematical statistics, the product of variances of two quantities is greater than or equal to the square of their covariance (Cramér, 1946b). In the following cases, it is equivalent to Schwarz inequality (28) with a suitable choice of the functions u and v.

First, we put

$$u = \Delta x \sqrt{\rho} \tag{48}$$

and

$$v = \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle\right)\sqrt{\rho}. \tag{49}$$

Then, the Schwarz inequality yields the first uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p_1)^2\rangle \geq \left[\int_{-\infty}^{\infty} \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle\right)\rho \, \mathrm{d}x\right]^2. \tag{50}$$

As it follows from section 5, the function $\partial s_1/\partial x$ in the last integral represents the momentum and this relation has the usual above mentioned meaning known from mathematical statistics. Depending on the functions s_1 and s_2, the square of the covariance of the coordinate and momentum at the right–hand side of this relation can have arbitrary values greater than or equal to zero.

The second uncertainty relation can be obtained in an analogous way for

$$u = \Delta x \sqrt{\rho} \tag{51}$$

and

$$v = \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle\right)\sqrt{\rho} \tag{52}$$

with the result

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle \geq \left[\int_{-\infty}^{\infty} \Delta x \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle\right)\rho \, \mathrm{d}x\right]^2. \tag{53}$$

It follows from Eq. (17) that the right–hand side of this relation can be simplified

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle \geq \left(\int_{-\infty}^{\infty} x \frac{\partial s_2}{\partial x} \rho \, \mathrm{d}x\right)^2. \tag{54}$$

Then, Eq. (23) leads to the final form of the second uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle \geq \frac{\hbar^2}{4}. \tag{55}$$

This uncertainty relation follows from the Schwarz inequality in a similar way as the first one, however, the covariance (u, v) is in this case constant and equals $\hbar/2 > 0$ independently of the concrete form of the functions s_2 or ρ. We note also that relation (55) is for $\langle x \rangle = a$ equivalent to inequality (33) for the Fisher information. It confirms again that inequality (33) is in quantum mechanics correctly respected.

Analogous uncertainty relations can be derived also in the multidimensional case (Skála & Kapsa, 2008; 2009) and for the mixed states described by the density matrix (Skála & Kapsa, 2009).
The sum of uncertainty relations (50) and (55) gives the relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \left[\int_{-\infty}^{\infty} \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle\right) \rho \, \mathrm{d}x\right]^2 + \frac{\hbar^2}{4}. \tag{56}$$

Heisenberg uncertainty relation (42) can be obtained from this relation by neglecting the first term on its right–hand side. Therefore, uncertainty relations (50) and (55) are stronger than the corresponding Heisenberg uncertainty relation (42).

10. Robertson–Schrödinger uncertainty relation

Relationship of uncertainty relations (50) and (55) to the Robertson–Schrödinger uncertainty relation (Peřinová et al., 1998; Robertson, 1929; 1934; Schrödinger, 1930a;b) can be clarified as follows (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2011; Skála, Čížek & Kapsa, 2011)).
For two linear hermitian operators $\hat{A}$ and $\hat{B}$, the Robertson–Schrödinger uncertainty relation can be written in the form

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}\left(\langle\{\Delta\hat{A}, \Delta\hat{B}\}\rangle^2 + \left|\langle[\hat{A}, \hat{B}]\rangle\right|^2\right), \tag{57}$$

where $\langle\hat{A}\rangle = \langle\psi|\hat{A}\psi\rangle$ is the mean value of the operator $\hat{A}$ in the state described by the wave function ψ, $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$, $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes the anticommutator and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ the commutator of the operators $\hat{A}$ and $\hat{B}$.
For the operators $\hat{x} = x$ and $\hat{p} = -i\hbar(\partial/\partial x)$ the straightforward calculation yields

$$\frac{1}{2}\langle\{\Delta x, \Delta\hat{p}\}\rangle = \frac{1}{2}\int_{-\infty}^{\infty} \psi^* \left[\Delta x\left(-i\hbar\frac{\partial}{\partial x} - \langle\hat{p}\rangle\right) + \left(-i\hbar\frac{\partial}{\partial x} - \langle\hat{p}\rangle\right)\Delta x\right]\psi \, \mathrm{d}x = \tag{58}$$

$$= \int_{-\infty}^{\infty} \Delta x\left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)\rho \, \mathrm{d}x.$$

Further, taking into account the commutation relation $[x, \hat{p}] = i\hbar$, relation (57) leads to Eq. (56). Therefore, relations (50) and (55) are stronger than both the Heisenberg and Robertson–Schrödinger relations (42) and (56) and yield more detailed information in terms of the mean square deviations $\langle(\Delta x)^2\rangle$, $\langle(\Delta p_1)^2\rangle$ and $\langle(\Delta p_2)^2\rangle$.
For the momentum represented by the function $p = \partial s_1/\partial x$, the mean value $\langle[\Delta x, \Delta p]\rangle$ equals zero and the Heisenberg and Robertson–Schrödinger uncertainty relations (42) and (56) do not contain the term $\hbar^2/4$. It shows again that this representation of the momentum is not, except for the cases discussed in sections 4 and 5, correct.
The equality sign in Schwarz inequality (28) is obtained if the functions u and v are collinear, i.e. for $u = const\, v$, where $const$ is a complex number. However, since the functions s_1, s_2 and ρ are real, the corresponding functions u and v are also real. Therefore, $const$ must be a real number or a real function of t. It follows from the conditions $u = const\, v$ for the functions s_1 and s_2 that these functions have to be quadratic functions of x of the form $p(t)x^2 + q(t)x + r(t)$, where real coefficients $p(t)$, $q(t)$ and $r(t)$ can depend on time.
It is worth to notice that the condition for the equality sign in relation (55) is independent of the form of the function s_1. Therefore, the equality sign in this relation can be achieved for much

larger class of the wave functions than in case of the Heisenberg or Robertson–Schrödinger uncertainty relations. It is interesting not only from the theoretical point of view but also from the point of view of some applications.

11. Free particle

In this section, we discuss uncertainty relations (42), (50), (55) and (56) in case of a free particle (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2011; Skála, Čížek & Kapsa, 2011)).
We assume that the wave function of a free particle is at time $t = 0$ described by the gaussian wave packet

$$\psi(x,0) = \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-x^2/(2a^2)+ikx} \tag{59}$$

with the energy

$$E = \frac{\hbar^2}{4ma^2} + \frac{\hbar^2 k^2}{2m}, \tag{60}$$

where $a > 0$ and k are real constants. By solving the time Schrödinger equation we get

$$\psi(x,t) = \frac{1}{\sqrt{a\sqrt{\pi}}} \frac{\sqrt{1 - \frac{i\hbar t}{ma^2}}}{\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}} \times \tag{61}$$

$$\times \exp\left\{ -\frac{\left(x - \frac{\hbar k}{m} t\right)^2}{2a^2\left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} + i\left[\frac{kx + \frac{\hbar t x^2}{2ma^4} - \frac{\hbar k^2}{2m} t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2}\right]\right\}.$$

The corresponding functions s_1 and s_2 and their derivatives equal

$$s_1(x,t) = \hbar k \frac{x + \frac{\hbar t x^2}{2ma^4 k} - \frac{\hbar k}{2m} t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2} - \hbar \arctan \frac{\hbar t}{ma^2}, \tag{62}$$

$$s_2(x,t) = \frac{\hbar}{2}\left\{ \frac{\left(x - \frac{\hbar k}{m} t\right)^2}{a^2\left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} - \ln \frac{1}{a\sqrt{\pi}\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}} \right\} \tag{63}$$

and

$$\frac{\partial s_1}{\partial x} = \hbar k \frac{1 + \frac{\hbar t x}{ma^4 k}}{1 + \left(\frac{\hbar t}{ma^2}\right)^2}, \tag{64}$$

$$\frac{\partial s_2}{\partial x} = \frac{\hbar\left(x - \frac{\hbar k}{m} t\right)}{a^2\left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]}. \tag{65}$$

As it could be anticipated, the mean momentum and the mean coordinate equal

$$\langle \hat{p} \rangle = \left\langle \frac{\partial s_1}{\partial x} \right\rangle = \hbar k \tag{66}$$

and

$$\langle x \rangle = \frac{\hbar k}{m} t. \tag{67}$$

The mean square deviations of the coordinate and momentum are given by the equations

$$\langle (\Delta x)^2 \rangle = \frac{a^2}{2} \left[1 + \left(\frac{\hbar t}{m a^2} \right)^2 \right], \tag{68}$$

$$\langle (\Delta p_1)^2 \rangle = \frac{\hbar^4 t^2}{2 m^2 a^6 \left[1 + \left(\frac{\hbar t}{m a^2} \right)^2 \right]} \tag{69}$$

and

$$\langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{2 a^2 \left[1 + \left(\frac{\hbar t}{m a^2} \right)^2 \right]}. \tag{70}$$

The left–hand side of relation (50) equals

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_1)^2 \rangle = \frac{\hbar^4 t^2}{4 m^2 a^4}. \tag{71}$$

Calculating the right–hand side of this relation we get the same result

$$\left\langle \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) \right\rangle^2 = \frac{\hbar^4 t^2}{4 m^2 a^4}. \tag{72}$$

Therefore, uncertainty relation (50) is fulfilled with the equality sign.
Calculating the left–hand side of uncertainty relation (55) we obtain

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{4} \tag{73}$$

and see that uncertainty relation (55) is fulfilled with the equality sign, too.
The corresponding Robertson–Schrödinger uncertainty relation has the form

$$\langle (\Delta x)^2 \langle (\Delta p)^2 \rangle \rangle] = \frac{\hbar^4 t^2}{4 m^2 a^4} + \frac{\hbar^2}{4} \tag{74}$$

and is fulfilled with the equality sign for all $t \geq 0$. The Heisenberg uncertainty relation (42) for our wave packet can be obtained if the first term on the right–hand side of the last equation is neglected.

12. Linear harmonic oscillator

The second example of application of uncertainty relations (50) and (55) is the linear harmonic oscillator in the coherent state described at time $t = 0$ by the gaussian wave packet (Skála, Kapsa & Lužová, 2011)

$$\psi(x, 0) = \left(\frac{m\omega}{\hbar \pi} \right)^{1/4} \mathrm{e}^{-(\xi - \xi_0)^2/2}, \tag{75}$$

where

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x, \tag{76}$$

$$\xi_0 = \sqrt{\frac{m\omega}{\hbar}}x_0 \tag{77}$$

and x_0 is the center of the packet. The corresponding energy E equals

$$E = \frac{m\omega^2 x_0^2}{2} + \frac{\hbar\omega}{2}. \tag{78}$$

By solving the time Schrödinger equation we get

$$\psi(x,t) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \mathrm{e}^{-i\omega t/2} \mathrm{e}^{i(m\omega/\hbar)[x_0^2\cos(\omega t) - 2xx_0]\sin(\omega t)/2} \mathrm{e}^{-(m\omega/\hbar)[x - x_0\cos(\omega t)]^2/2}. \tag{79}$$

The corresponding functions s_1 and s_2 equal

$$s_1(x,t) = -\hbar\omega t/2 + (m\omega)[x_0^2\cos(\omega t) - 2xx_0]\sin(\omega t)/2 \tag{80}$$

and

$$s_2(x,t) = \frac{\hbar}{4}(\ln\hbar + \ln\pi - \ln m - \ln\omega) + \frac{m\omega}{2}[x - x_0\cos(\omega t)]^2. \tag{81}$$

The mean momentum and the mean coordinate have the same form as in classical mechanics

$$\langle\hat{p}\rangle = \left\langle\frac{\partial s_1}{\partial x}\right\rangle = -m\omega x_0\sin(\omega t) \tag{82}$$

and

$$\langle x\rangle = x_0\cos(\omega t). \tag{83}$$

The mean square deviations of the coordinate and momentum from their mean values are given by the equations

$$\langle(\Delta x)^2\rangle = \frac{\hbar}{2m\omega}, \tag{84}$$

$$\langle(\Delta p_1)^2\rangle = 0 \tag{85}$$

and

$$\langle(\Delta p_2)^2\rangle = \frac{\hbar m\omega}{2}. \tag{86}$$

It means that uncertainty relations (50) and (55) have the form

$$0 = 0 \tag{87}$$

and

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle = \frac{\hbar^2}{4}. \tag{88}$$

It is seen that uncertainty relation (50) has in this case very simple form $0 = 0$. It follows from equation (88) that the left–hand side of relation (55) achieves for this example its minimum $\hbar^2/4$.

13. Commutation relations and vector potential

To introduce potentials, we make use of Eq. (27) with $a = 0$

$$\int_{-\infty}^{\infty} x \frac{\partial \rho}{\partial x} \, \mathrm{d}x = -1. \tag{89}$$

Using Eq. (14) we get

$$\int_{-\infty}^{\infty} x \left(\frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right) \mathrm{d}x = -1. \tag{90}$$

Performing integration by parts in the first term and taking into account conditions (6) we have

$$\int_{-\infty}^{\infty} \left[\psi^* x \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x} (x\psi) \right] \mathrm{d}x = -1. \tag{91}$$

Multiplying this equation by $-i\hbar$ we obtain the equation

$$\int_{-\infty}^{\infty} \psi^* [x, \hat{p}] \psi \, \mathrm{d}x = i\hbar \tag{92}$$

indicating validity of the commutation relation

$$[x, \hat{p}] = i\hbar \tag{93}$$

known from quantum mechanics.
Further, it is seen that Eq. (92) is valid also in case that the momentum operator $\hat{p}$ is replaced by $\hat{p} - f$, where $f = f(x, t)$ is a real function. This function can describe external conditions in which the system moves. In physics, such functions are usually denoted as the vector potential. For example, the function f can equal qA in Eq. (35) (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2005a;b; 2007a)).

14. Time

Systems investigated in standard quantum mechanics are supposed to have infinite lifetime. Therefore, normalization condition (5) is for such systems valid at all times t from the preparation of the system in a state described by ψ at time $t = t_1$ to the subsequent measurement at later time t_2. Therefore, the probability to find the measured system anywhere in space equals one for all times $t_1 \leq t \leq t_2$. For this reason, it does not make sense to introduce the probability density in time analogous to the probability density in space and time is taken as a parameter in standard quantum mechanics.
Rather different situation is obtained if we assume that the investigated state has a finite lifetime and the probability to find the system anywhere in space

$$\nu(t) = \int_{-\infty}^{\infty} \rho(x, t) \, \mathrm{d}x \tag{94}$$

decays in time (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2005a;b; 2007a)). Normalization of ν is given by the equation

$$\int_{t_1}^{\infty} \nu \, \mathrm{d}t = 1 \tag{95}$$

expressing the fact that, after its preparation at time $t = t_1$, the investigated state decays with the probability equal to one. This generalization includes standard quantum mechanics with the infinite lifetime as a limit case.
By analogy with the coordinate x, it is then possible to define the mean lifetime τ

$$\tau = \langle t - t_1 \rangle = \int_{t_1}^{\infty} (t - t_1) \nu \, \mathrm{d}t, \tag{96}$$

the mean value of the square of time

$$\langle (t - t_1)^2 \rangle = \int_{t_1}^{\infty} (t - t_1)^2 \nu \, \mathrm{d}t \tag{97}$$

and the corresponding mean square deviation

$$\langle [\Delta(t - t_1)]^2 \rangle = \langle (t - t_1)^2 \rangle - \langle t - t_1 \rangle^2. \tag{98}$$

15. Scalar potential

Similarly to Eq. (26), we perform integration by parts with respect to time in Eq. (95) and get

$$(t - t_1)\nu\big|_{t=t_1}^{\infty} - \int_{t_1}^{\infty} (t - t_1) \frac{\mathrm{d}\nu}{\mathrm{d}t} \mathrm{d}t = 1. \tag{99}$$

By analogy with Eq. (6) we can assume validity of conditions

$$\lim_{t \to t_1} (t - t_1)^n \nu = 0, \quad n = 0, 1, 2 \tag{100}$$

and

$$\lim_{t \to \infty} (t - t_1)^n \nu = 0, \quad n = 0, 1, 2. \tag{101}$$

Using Eqs. (14), (94), (100) and (101) we get from Eq. (99)

$$\int_{t_1}^{\infty} (t - t_1) \left[\int_{-\infty}^{\infty} \left(\frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t} \right) \mathrm{d}x \right] \mathrm{d}t = -1. \tag{102}$$

Performing integration by parts in the first term and taking into account Eqs. (100) and (101) we have

$$\int_{t_1}^{\infty} \int_{-\infty}^{\infty} \left\{ \psi^* (t - t_1) \frac{\partial \psi}{\partial t} - \psi^* \frac{\partial}{\partial t} [(t - t_1)\psi] \right\} \mathrm{d}x \, \mathrm{d}t = -1. \tag{103}$$

Multiplying this equation by $-i\hbar$ we obtain the equation

$$\int_{t_1}^{\infty} \int_{-\infty}^{\infty} \psi^* \left[i\hbar \frac{\partial}{\partial t}, t - t_1 \right] \psi \, \mathrm{d}x \, \mathrm{d}t = i\hbar. \tag{104}$$

This result indicates that for systems with a finite lifetime the operator $i\hbar(\partial/\partial t)$ has analogous mathematical properties as the momentum operator $\hat{p}$.
Further, it is seen that Eq. (99) remains valid even in case when the operator $i\hbar(\partial/\partial t)$ is replaced by the operator $i\hbar(\partial/\partial t) - g$, where $g = g(x, t)$ is a real function. Analogously to the function f, the function g can describe external conditions in which the system moves. For example, the function g can equal qV, where q is the charge and V the scalar potential of the electromagnetic field (see also (Skála & Kapsa, 2005a;b; 2007a)).

16. Time–energy uncertainty relations

To derive the "time–energy" uncertainty relation, we start from the equation that is a bit more general than Eq. (99) and is analogous to Eq. (27) (see also (Skála & Kapsa, 2007a))

$$\int_{t_1}^{\infty} (t - t_1 - \langle t - t_1 \rangle) \frac{\mathrm{d}\nu}{\mathrm{d}t} \mathrm{d}t = -1. \tag{105}$$

By using Schwarz inequality (28) for

$$u = \Delta(t - t_1)\sqrt{\nu}, \tag{106}$$

where

$$\Delta(t - t_1) = t - t_1 - \langle t - t_1 \rangle, \tag{107}$$

and

$$v = \frac{1}{\sqrt{\nu}} \frac{\mathrm{d}\nu}{\mathrm{d}t} \tag{108}$$

we get the inequality

$$\int_{t_1}^{\infty} [\Delta(t - t_1)]^2 \nu \,\mathrm{d}t \int_{t_1}^{\infty} \frac{1}{\nu} \left(\frac{\partial \nu}{\partial t} \right)^2 \mathrm{d}t \geq 1 \tag{109}$$

analogous to inequality (31). It is a general form of the "time–energy" uncertainty relation. As an example, we assume now that the probability $\nu(t)$ to find the system in state ψ decays exponentially in time

$$\nu(t) = \frac{1}{\tau} \mathrm{e}^{-(t-t_1)/\tau}, \tag{110}$$

where τ denotes the lifetime. The corresponding mean values $\langle t - t_1 \rangle$, $\langle (t - t_1)^2 \rangle$ and $\langle [\Delta(t - t_1)]^2 \rangle$ equal

$$\langle t - t_1 \rangle = \int_{t_1}^{\infty} (t - t_1) \nu \,\mathrm{d}t = \tau, \tag{111}$$

$$\langle (t - t_1)^2 \rangle = \int_{t_1}^{\infty} (t - t_1)^2 \nu \,\mathrm{d}t = 2\tau^2 \tag{112}$$

and

$$\langle [\Delta(t - t_1)]^2 \rangle = \int_{t_1}^{\infty} [\Delta(t - t_1)]^2 \nu \,\mathrm{d}t = \langle (t - t_1)^2 \rangle - \langle t - t_1 \rangle^2 = \tau^2. \tag{113}$$

Further, we assume that the wave function describing the state with a finite lifetime has the following simple form

$$\psi(x, t) = \sqrt{\frac{2E_2}{\hbar}} \mathrm{e}^{(E_1 - iE_2)(t - t_1)/(i\hbar)} \psi_0(x), \tag{114}$$

where E_1 and $E_2 > 0$ are the real and imaginary part of the energy, respectively, and $\psi_0(x)$ is the space part of the wave function. Then, using Eq. (114), we calculate the second integral in Eq. (109) and get

$$\int_{t_1}^{\infty} \frac{1}{\nu} \left(\frac{\partial \nu}{\partial t} \right)^2 \mathrm{d}t = \frac{4E_2^2}{\hbar^2}. \tag{115}$$

The resulting time–energy uncertainty relation has the form

$$\tau^2 E_2^2 \geq \frac{\hbar^2}{4}. \tag{116}$$

This relation shows that the lifetime and imaginary part of the energy are not independent and obey the well–known time–energy uncertainty relation.
To determine the shape of the corresponding spectral line it is necessary to calculate the Fourier transform of the function (110). As a result, the Lorentz form of the spectral line is obtained.

17. Equations of motion

As mentioned above, to describe motion in space both the probability density ρ and probability density current j or the functions s_1 and s_2 have to be used. To describe time evolution, integration in the Fisher information should be obviously performed not only over the space coordinates but also over time. For these reasons and in agreement with the last three sections, we define a generalized space Fisher information in the form (see also (Kapsa & Skála, 2011; Skála & Kapsa, 2005a;b; 2007a))

$$J_x = \frac{4}{\hbar^2}\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial s_1}{\partial x}\right)^2+\left(\frac{\partial s_2}{\partial x}\right)^2\right]\rho\,\mathrm{d}x\,\mathrm{d}t = 4\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial\psi}{\partial x}\right|^2\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{117}$$

Analogously, we define a generalized time Fisher information

$$J_t = \frac{4}{\hbar^2}\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial s_1}{\partial t}\right)^2+\left(\frac{\partial s_2}{\partial t}\right)^2\right]\rho\,\mathrm{d}x\,\mathrm{d}t = 4\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial\psi}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t \geq 0, \tag{118}$$

where $\psi = \exp[(is_1 - s_2)/\hbar]$ and $\rho = |\psi|^2$. Both generalized Fisher informations depend on the space and time derivatives of the functions s_1 and s_2 in a similar way. Since there are no potentials in the last two equations, they correspond to a free motion.
To find equations of motion, we need an additional physical principle. To describe physical phenomena in a way independent of the choice of the concrete coordinate system and the state of the investigated system, we require that the combined generalized space–time Fisher information equals a real constant *const*

$$\frac{J_t}{c^2} \pm J_x = const. \tag{119}$$

Here, c is the speed of light and the sign in front of the generalized spatial Fisher information J_x can be either plus or minus.
First we notice that the space initial conditions for the wave function ψ at $t = 0$ can be from the mathematical point of view chosen arbitrarily and J_x can have arbitrary values greater than or equal to zero. In contrast to it, the wave function ψ at later times is given by the evolution consistent with Eq. (119). It makes possible to derive the equation of motion from this equation.
Further, to determine the sign in Eq. (119), we consider a free particle which is at rest in a given coordinate system. It follows from Eq. (38) with $A = 0$ that it is obtained for very small values of $|\partial s_1/\partial x|$ and $|\partial s_2/\partial x|$. In such a case, the Fisher information J_x is close to zero and Eq. (119) yields

$$const \geq 0. \tag{120}$$

Then, we consider a particle having a large kinetic energy T and a large Fisher information $J_x > const$. In such a case, it is impossible to obey Eq. (119) with the plus sign. Therefore, we can conclude that the sign in Eq. (119) must be negative

$$\frac{J_t}{c^2} - J_x = const. \tag{121}$$

It is seen that this combination of the Fisher informations J_t and J_x is Lorentz invariant.
The last equation can be rewritten into the form

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{c^2}\left|\frac{\partial\psi}{\partial t}\right|^2 - \left|\frac{\partial\psi}{\partial x}\right|^2 - \frac{const}{4}|\psi|^2\right)\mathrm{d}x\,\mathrm{d}t = 0. \tag{122}$$

This equation must be valid for arbitrary initial conditions at $t = t_1$, i.e., it has to be independent of ψ. Therefore, its variation must equal zero

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{c^2}\frac{\partial\delta\psi^*}{\partial t}\frac{\partial\psi}{\partial t} - \frac{\partial\delta\psi^*}{\partial x}\frac{\partial\psi}{\partial x} - \frac{const}{4}\delta\psi^*\psi\right)\mathrm{d}x\,\mathrm{d}t + c.c. = 0, \tag{123}$$

where $\delta\psi$ denotes the variation of ψ. Performing integration by parts with respect to t in the first term and with respect to x in the second one and assuming that variations $\delta\psi$ and $\delta\psi^*$ equal zero at the borders of the integration region we get

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\delta\psi^*\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{const}{4}\right)\psi\,\mathrm{d}x\,\mathrm{d}t + c.c. = 0. \tag{124}$$

This equation has to be obeyed for arbitrary values of $\delta\psi$ and $\delta\psi^*$. It leads to the equation of motion

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{const}{4}\right)\psi = 0. \tag{125}$$

We see that except for the number of space dimensions and the constant *const*, this equation has the same mathematical form as the Klein–Gordon equation known from quantum mechanics

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0, \tag{126}$$

where m_0 is the rest mass.
We note that another derivation of the Klein–Gordon equation is given in (Frieden, 1998; 2004; Frieden & Soffer, 1995; Reginatto, 1998; 1999).
As it is known, the Schrödinger equation for a free particle

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m_0}\Delta\varphi \tag{127}$$

can be obtained from the Klein–Gordon equation in the non–relativistic approximation for the function

$$\psi = \mathrm{e}^{m_0c^2t/(i\hbar)}\varphi, \tag{128}$$

where φ is the wave function appearing in the Schrödinger equation.
The Dirac equation for a free particle can be also obtained in a similar way (see also (Frieden, 1998; 2004; Frieden & Soffer, 1995; Skála & Kapsa, 2005a;b; 2007a)).
Potentials can be included into the theory by the method described in sections 13 and 15.
It worth to notice that the equations of motion discussed above are linear and the superposition principle is for them valid. This property can be traced back to the expression

(32) for the Fisher informations I_x. By using the substitution $\rho = \exp(-2s_2/\hbar)$, I_x can be written in terms of the square of the function $\partial s_2/\partial x$ (see Eq. (34)). Similar approach is used in Eqs. (117) and (118) for J_x and J_t, too. Then, using Eq. (121) and performing the variations and integration by parts in Eq. (122), the squares of the functions $\partial\psi/\partial x$ and $\partial\psi/\partial t$ disappear and the second partial derivatives of ψ with respect to the coordinates and time are obtained. Therefore, the resulting equations of motion are linear.

The role of the operator $i\hbar(\partial/\partial t)$ is different from the role of the energy operator — hamiltonian. In agreement with discussion in this section, the operator $i\hbar(\partial/\partial t)$ is important for describing the time evolution of the wave function given by the equations of motion.

We note also that condition (121) of the constant value of the generalized space–time Fisher information expressed in the variational form yields in the limit of classical mechanics the Hamilton principle (Kapsa & Skála, 2009).

Finally we note that quantization known from quantum mechanics is consequence of the statistical description of results of measurement and boundary conditions applied to the wave function ψ. As it is known, only some solutions of equations of motion obey these conditions and possible states of quantum systems can be quantized.

18. Conclusion

We have shown that the basic mathematical structure of quantum mechanics can be understood as generalization of classical mechanics in which the statistical character of results of measurement is taken into account and the most important general properties of statistical theories known from mathematical statistics are correctly respected. It is not therefore surprising that quantum mechanics yields correct description of physical reality in agreement with experiments.

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